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Implementing the core of a two-person pure allocation game without free disposal or integer games

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We show that implementation of the core of a two-person pure allocation game can be achieved without the use of 'free disposal', 'integer games', or any refinement of Nash equilibria.

1. Introduction and motivation

A class of Nash implementation problems of particular economic interest is that of implementing the core in games of pure allocation, or 'cake division'. Mechanisms that have been proposed to solve this type of problem often involve off-equilibrium 'punishments' that lie in the interior of the feasible set; that is, in which not all the cake is allocated. They also tend to make use of extraneous devices such as 'integer games'. Moreover, many mechanisms rely on there being three or more players.

This note looks at the problem of implementing the core in a *two-person* 'Edgeworth Box' economy *without free disposal and without integer games*. The mechanism we propose, in or out of equilibrium, only involves allocations that lie inside the box. Agents' strategy sets are limited to proposing an allocation and announcing a preference propfile.

Free disposal can be viewed as allowing for a dummy player to act as a 'sink' for surplus, hence our restriction might be more in the spirit of a genuine two player problem. Moreover, free disposal may be an inappropriate assumption in some contexts, regardless of the number of players. For example, consider the allocation of hazardous waste. Here 'free disposal' is the threat of generating more waste and may mechanism that threatens this may not be 'credible' or 'renegotiation proof'.

The idea behind the 'punishment set' used by our mechanism in place of free disposal has been generalised by Moore and Repullo (1990) and by Dutta and Sen (1991). The implementation games that they suggest, however, use 'integer games'.

The main argument against the use of 'integer games' is that they add complexity to the agents' problem. Not only are they somewhat extraneous, but they also result in unbounded strategy sets. Jackson and Palfrey (1990) and Sjöström (1990) have looked at implementation in bounded mechanisms in a general context, but they allow 'free disposal'. Moreover, both these papers concern *undominated* Nash implementation. In general, refining the equilibrium concept makes implementation easier.

The mechanisms we propose, exploits a property of the social choice correspondence itself (specifically, Pareto optimality) to avoid having to use 'integer games'. While our mechanism is designed for a specific case, perhaps this 'trick' might throw some light on more general problems.

2. Notation and assumptions

2.1. An Edgeworth Box

Consider an 'Edgeworth Box' economy in L divisible goods. Let $\bar{A}_j$ be the total endowment of good j, and $\boldsymbol{a}_j$ be player 1's final allocation of good j; then the set of possible final allocations to player 1 is $A = \{\boldsymbol{a} \in \mathbb{R}^L \mid \forall j,\ \boldsymbol{a}_j \in [0, \bar{A}_j]\}$. The assumption that everything must be allocated implies that the final allocation of good j to player 2 is given by $\bar{A}_j - \boldsymbol{a}_j$. Since the game is 'constant sum in each good', we need only to refer explicitly to player 1's allocation. For example, we shall denote player 1's endowment by $\boldsymbol{e}$.

2.2. Restrictions on preferences

Let Θ_1 (resp. Θ_2) be the set of preference relations over A that are complete, transitive, reflexive, continuous, strictly quasi-concave, and locally non-satiated. Let (Θ_1, Θ_2) denote an element of $\Theta_1 \times \Theta_2$. Note that our restrictions rule out 'flat' indifference sets and imply that weak and strong Pareto optimality are equivalent.

2.3. Restrictions on endowments

We will assume that $\boldsymbol{e}$ is in the interior of A and that $\boldsymbol{e}$ is observable so the implementation mechanism can be contingent on $\boldsymbol{e}$.

2.4. Notation

For $i = 1, 2$, let: $L(\theta_i, \boldsymbol{a}) = \{\boldsymbol{a}' \in A \mid \boldsymbol{a} \succsim_{\theta_i} \boldsymbol{a}'\}$,
and: $U(\theta_i, \boldsymbol{a}) = \{\boldsymbol{a}' \in A \mid \boldsymbol{a}' \succ_{\theta_i} \boldsymbol{a}\}$.
These are closed lower and open upper contour sets defined for particular allocations and preferences.

For any allocation $\boldsymbol{a}$, let $D(\theta_1, \theta_2, \boldsymbol{a})$ be the set of feasible allocations that (strictly) Pareto dominate a according to this preference profile; that is:

$$D(\theta_1, \theta_2, \boldsymbol{a}) = \{\boldsymbol{a}' \in A \mid \boldsymbol{a}' \succ_{\theta_1} \boldsymbol{a}, \text{ and } \boldsymbol{a}' \succ_{\theta_2} \boldsymbol{a}\}.$$

Let $C(\theta_1, \theta_2)$ denote the 'contract curve' in the Edgeworth box, A, for the preference profile (θ_1, θ_2); that is:

$$C(\theta_1, \theta_2) = \{\boldsymbol{a}' \in A \mid D(\theta_1, \theta_2, \boldsymbol{a}') = \emptyset\}.$$

3. The social choice correspondence and the mechanism

3.1. The social choice correspondence

Let (Σ_1, Σ_2) be the agents' true preference profile over allocations. The social choice rule is given by:

$$h(\Sigma_1, \Sigma_2, e) = \{a \in A \mid a \gtrsim_{\Sigma_1} e,\ a \gtrsim_{\Sigma_2} e, \text{ and } D(\Sigma_1, \Sigma_2, a) = \emptyset\}.$$

3.2. The mechanism strategies

The players' strategies in the implementation game are as follows. The players simultaneously announce a preference profile for both agents and an allocation that lies in the social choice correspondence for these preferences. Formally, let:
$s_1 = (\xi_1, \xi_2, a)$ be the preference profile and the allocation announced by player one, where $(\xi_1, \xi_2) \in \Theta_1 \times \Theta_2$ and $a \in h(\xi_1, \xi_2, e)$. Similarly, let: $s_2 = (\phi_1, \phi_2, b)$ be the preference profile and the allocation announced by player two, where $(\phi_1, \phi_2) \in \Theta_1 \times \Theta_2$ and $b \in h(\phi_1, \phi_2, e)$.

3.3. 'Punishments'

We need a way to 'punish' players out of equilibrium. Let P: $\Theta_1 \times \Theta_2 \times A \to A$ be a function such that:

$$P(\phi_1, \xi_2, e) \in \text{Int}(L(\phi_1, e) \cap L(\xi_2, e)).$$

The idea of the punishment is that it gives to each player an allocation that is strictly worse than the endowment, according to the preferences ascribed to him *by the other player*. The function P can be defined since the strict quasiconcavity and local nonsatiation of preference ensures that $\text{Int}(L(\phi_1, e) \cap L(\xi_2, e))$ is not empty. Note that, by the definition of a and b,

$$P(\phi_1, \xi_2, e) \in \text{Int}(L(\phi_1, b) \cap L(\xi_2, a)).$$

3.4. The allocation rule

The mechanism assigns allocations to strategy combinations by the following algorithm.

(I)	IF	$(\xi_1, \xi_2) = (\phi_1, \emptyset_2)$	
	AND	$a = b$	THEN: IMPLEMENT a.
ELSE (II.i)	IF	$a \in L(\phi_1, b) \cap C(\phi_1, \xi_2)$	
	AND	$b \notin L(\xi_2, a) \cap C(\phi_1, \xi_2)$	THEN: IMPLEMENT a.
ELSE (II.ii)	IF	$b \in L(\xi_2, a) \cap C(\phi_1, \xi_2)$	
	AND	$a \notin L(\phi_1, b) \cap C(\phi_1, \xi_2)$	THEN: IMPLEMENT b.
ELSE (III)			IMPLEMENT $P(\phi_1, \xi_2, e)$.

3.5. Observations

Note that, if either allocation rule (II.i) or (II.ii) apply then $\boldsymbol{a} \neq \boldsymbol{b}$.

The proposition below depends on the following fact. Allocation rule (II.i) enables player 1, given player 2's strategy, to obtain any allocation in A that is 'no better' for him than the allocation suggested by player 2 according to the preferences assigned to him by player 2; i.e., the whole of $\boldsymbol{L}(\phi_1, \boldsymbol{b})$. Player 1 does this by announcing $(\xi_1, \xi_2, \boldsymbol{a}')$ where ξ_1 and ξ_2 are chosen such that $\boldsymbol{a}'$ is an element of $C(\phi_1, \xi_2)$ and such that $\boldsymbol{b}$ is not an element of $\boldsymbol{L}(\xi_2, \boldsymbol{a}) \cap \boldsymbol{C}(\phi_1, \xi_2)$. Similarly for player 2 through (II.ii). Therefore,

Definition. We will refer to the set $\boldsymbol{L}(\phi_1, \boldsymbol{b})$ (resp. $\boldsymbol{L}(\xi_2, \boldsymbol{a})$) as obtainable by player 1 (resp. 2).

4. Proposition

Proposition. Given that $(\Sigma_1, \Sigma_2) \in \Theta_1 \times \Theta_2$, *and* $\boldsymbol{e} \in \text{Int}(A)$, *the above mechanism implements* $\boldsymbol{h}(\Sigma_1, \Sigma_2, \boldsymbol{e})$ *as Nash equilibria.*

Proof. It suffices to show that all allocations in $\boldsymbol{h}(\Sigma_1, \Sigma_2\ \boldsymbol{e})$ can be supported as equilibrium allocations, and that all equilibrium allocations are in $\boldsymbol{h}(\Sigma_1, \Sigma_2, \boldsymbol{e})$.

Step 1.

Claim: for any $\boldsymbol{a}^* \in \boldsymbol{h}(\Sigma_1, \Sigma_2, \boldsymbol{e})$, there exist some Nash equilibrium strategies such that $\boldsymbol{a}$ is the resulting allocation.

Consider the strategies $s_1 = s_2 = (\Sigma_1, \Sigma_2, \boldsymbol{a}^*)$. The allocations obtainable by player 1 by unilateral deviation are $\boldsymbol{L}(\Sigma_1, \boldsymbol{a}^*)$. But, since player 2's announcement of player 1's preferences are the true preferences, Σ_1, player 1 can not strictly prefer these obtainable allocations $\boldsymbol{a}$. Similarly for player 2.

Step 2.

Claim: If $s_1 = s_2 = (\hat{\theta}_1, \hat{\theta}_2, \boldsymbol{a}^*)$ form an equilibrium but $(\hat{\theta}_1, \hat{\theta}_2) \neq (\Sigma_1, \Sigma_2)$, then $\boldsymbol{a}^* \in \boldsymbol{h}(\Sigma_1, \Sigma_2, \boldsymbol{e})$.

Assume (contra-hypothesis) that $\boldsymbol{a}^* \neq \boldsymbol{h}(\Sigma_1, \Sigma_2, \boldsymbol{e})$, then (i) $\boldsymbol{e} >_{\Sigma_i}$ for $i = 1$ or 2; or (ii) $\boldsymbol{D}(\sigma_1, \Sigma_2, \boldsymbol{a}^*) \neq \emptyset$. But if (i), $\boldsymbol{e}$ is obtainable by either player. If (ii), then since $s_1 = s_2$, it follows that for all $\boldsymbol{a}' \in A$:

$$\boldsymbol{a}' \in \boldsymbol{L}(\hat{\theta}_1, \boldsymbol{a}^*) \cup \boldsymbol{L}(\hat{\theta}_2, \boldsymbol{a}^*).$$

Thus $\boldsymbol{a}'$ is obtainable by player 1 and/or player 2. Hence, in particular, for any $\boldsymbol{d} \in \boldsymbol{D}(\Sigma_1, \Sigma_2, \boldsymbol{a}^*)$, $\boldsymbol{d}$ is obtainable by player 1 and/or player 2. Therefore (s_1, s_2) cannot be an equilibrium. □

Step 3.i.

Claim: If $(\xi_1, \xi_2, \boldsymbol{a})$ and $(\phi_1, \phi_2, \boldsymbol{b})$ form an equilibrium and allocation rule (II.i) applies [i.e., $\boldsymbol{a} \in \boldsymbol{L}(\phi_1, \boldsymbol{b}) \cap C(\phi_1, \xi_2)$ and $\boldsymbol{b} \notin \boldsymbol{L}(\xi_2, \boldsymbol{a}) \cap \text{C}(\phi_1, \xi_2)$] then $\boldsymbol{a} \in \boldsymbol{h}(\Sigma_1, \Sigma_2, \boldsymbol{e})$.

Assume (contra-hypothesis) that $\boldsymbol{a} \in \boldsymbol{h}(\Sigma_1, \Sigma_2, \boldsymbol{e})$, then (i) $\boldsymbol{e} >_{\Sigma_i} \boldsymbol{a}$ for $i = 1$ or 2; or (ii) $\boldsymbol{D}(\Sigma_1, \Sigma_2, \boldsymbol{a}) \neq \emptyset$. But, if (i) then $\boldsymbol{e}$ is obtainable by either player. If (ii), then since $\boldsymbol{e} \in \boldsymbol{h}(\xi_1, \xi_2, \boldsymbol{e})$ it follows that, for all $\boldsymbol{d} \in \boldsymbol{D}(\Sigma_1, \Sigma_2, \boldsymbol{a})$:

$$\boldsymbol{d} \in \boldsymbol{L}(\xi_1, \boldsymbol{a}) \cup \boldsymbol{L}(\xi_2, \boldsymbol{a}).$$

If $\boldsymbol{d} \in \boldsymbol{L}(\xi_2, \boldsymbol{a})$ then it is obtainable by player 2. If $\boldsymbol{d} \notin \boldsymbol{L}(\xi_2, \boldsymbol{a})$, then $\boldsymbol{d} \in U(\xi_2, \boldsymbol{a})$. But, since $\boldsymbol{a} \in \boldsymbol{C}(\phi_1, \xi_2)$ and the preferences are strictly quasi-concave and locally non-satiated, $U(\xi_2, \boldsymbol{a}) \subset \boldsymbol{L}(\phi_1, \boldsymbol{a})$. Moreover, given allocation rule (II.i), $\boldsymbol{a} \in \boldsymbol{L}(\phi_1, \boldsymbol{b})$. Therefore $\boldsymbol{L}(\phi_1, \boldsymbol{a}) \subseteq \boldsymbol{L}(\phi_1, \boldsymbol{b})$. Thus $\boldsymbol{d}$ is obtainable by player 1. □

Step 3.ii.

Claim: if $(\xi_1, \xi_2, \boldsymbol{a})$ and $(\phi_1, \phi_2, \boldsymbol{b})$ form an equilibrium and allocation rule (II.ii) applies then $\boldsymbol{b} \in \boldsymbol{h}(\Sigma_1, \Sigma_2, \boldsymbol{e})$.

Proof. Similar to 3.i. □

Step 4.

Claim: Allocation rule (III) can not apply in equilibrium. By definition $\boldsymbol{P}(\phi_1, \xi_2, \boldsymbol{e})$ is in the interior of $\boldsymbol{L}(\phi_1, \boldsymbol{b}) \cap \boldsymbol{L}(\xi_2, \boldsymbol{a})$. Hence, by local nonsatiation, there exists a point, $\boldsymbol{a}' \in \boldsymbol{L}(\phi_1, \boldsymbol{b})$ (thus obtainable by player 1), such that $\boldsymbol{a}' >_{\Sigma_1} \boldsymbol{P}(\phi_1, \xi_2, \boldsymbol{e})$. ■

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