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A CARDINAL CHARACTERIZATION OF THE RUBINSTEIN-SAFRA-THOMSON AXIOMATIC BARGAINING THEORY

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1. INTRODUCTION

IN A RECENT PAPER Rubinstein, Safra, and Thomson[2] (hereafter, RST) have provided an interesting re-examination of the widely applied Nash solution for a two-person bargaining problem. They recast the usual Nash bargaining problem into a more "natural" setting of feasible alternatives with a disagreement outcome. The two players are then described by their risk *preferences* defined on the set of lotteries over the alternatives and the disagreement outcome. This enables them to define an *ordinal* Nash solution in terms of the agents' risk preferences. Essentially their ordinal solution is an outcome that is "immune" against possible objections.

Freeing the definition of the Nash solution from "utility" naturally led RST to extending its scope to non-expected utility preferences. We contend, however, that the family of non-expected utility preferences considered by RST is unduly restrictive. The assumptions imposed on the risk preferences by RST essentially exclude any members of the Rank Dependent Expected Utility (RDEU) and betweenness families that can accommodate the very choice paradoxes that stimulated the development of non-expected utility theory.[3] As these are two of the most extensively analyzed and widely applied non-expected utility models in the literature, this seems to cast doubt on how broad an extension to non-expected utility preferences the RST approach affords.[4]

We demonstrate, however, that RST's analysis can be modified so that their conclusion is valid in a wider class of preferences that can include examples of RDEU preferences. This class consists of preferences that admit what we term a *disagreement linear* representation. This essentially means that for the set of elementary lotteries, lotteries whose support consists of at most one outcome and the disagreement outcome, there exists an expected utility representation of the preference relation. More significantly, in accomplishing this extension we develop a *cardinal characterization* of the ordinal Nash solution, which enables us to reduce RST's analysis to a straightforward corollary of Nash's original theorem and provides us an operationally simple method with which to compute the ordinal Nash solution.

Despite our success in characterizing a broader class of preferences, we highlight that preferences from this class cannot accommodate the variant of the Allais Paradox referred to as the *common ratio paradox*. Moreover, we give a simple example of preferences which can accommodate this paradox and yet for which an ordinal Nash

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[2] Rubinstein-Safra-Thomson (1992).

[3] For a detailed discussion of the RDEU model, see Quiggin (1992); for the betweenness model, see Chew (1989).

[4] Valenciano and Zarzuelo (1994) provide the ordinal analog of the nonsymmetric bargaining solution. For their extension to non-expected utility, they accept without comment the assumptions on risk preferences imposed by RST, except to note that all of these assumptions are guaranteed for expected utility.

solution does not, in general, exist. We thus think it is an interesting open question if one can characterize a family of risk preferences that can accommodate at least the most widely observed violations of expected utility theory and for which the ordinal Nash solution (or perhaps some suitably defined variant) is well-defined.[5]

2. THE MODEL AND RESULTS

Following RST, we consider a two agent bargaining problem as follows. The set of alternatives is denoted by X. X is assumed to be a nonempty, compact metrizable space. There is a given, designated disagreement outcome, denoted by D. The set $X \cup \{D\}$ is endowed with the natural topology where the point D is regarded as a discrete point, and we consider the associated Borel σ-field. We denote the set of lotteries (probability measures) over X and $X \cup \{D\}$ by $\mathscr{L}(X)$ and $\mathscr{L}(X \cup \{D\})$, respectively, which are endowed with the weak topology. $\mathscr{L}(X)$ and $\mathscr{L}(X \cup \{D\})$ are separable metrizable spaces. Let δ_x denote a (degenerate) lottery whose support consists of a single outcome $x \in X \cup \{D\}$. Without loss of generality, we can write a lottery over $X \cup \{D\}$ as $pL + (1-p)\delta_D$, where L is a lottery whose support is contained in X and $p \in [0, 1]$, and we shall denote it by pL for simplicity. Abusing notation, we shall often write px instead of $p\delta_x$ for any outcome $x \in X$. We shall call a lottery of the form px an *elementary* lottery. The set of all elementary lotteries is denoted by $\mathscr{L}_e$.

There are two agents, 1, 2. Each agent i, $i = 1, 2$, has preferences $\succeq_i$ over $\mathscr{L}(X \cup \{D\})$. We assume that each $\succeq_i$ is a complete, reflexive, continuous pre-order, and outcome D is least desirable, that is, $pL \succeq_i D$ for any $p \in (0, 1]$ and $L \in \mathscr{L}(X)$. Moreover, there exists at least one outcome $x \in X$, such that $x \succ_i D$, for both i. The set X and the disagreement outcome D are kept fixed, so a bargaining problem is naturally identified with $(\succeq_1, \succeq_2)$. In order to ease exposition and avoid uninteresting multiplicity of bargaining outcomes that are equivalent in terms of the agents' preferences, again following RST, we restrict the set of bargaining problems considered to ones that satisfy the following assumption.

SIMPLIFYING ASSUMPTION: *$x \sim_i y$ for both i implies $x = y$. For each i there exists an alternative b_i, such that $b_i \sim_j D$ and $b_i \succ_i x$ for all $x \neq b_i$.*

We shall consider the following additional assumptions on preferences and additional restriction to the set of bargaining problems:

DOM (First Order Stochastic Dominance): *If $x \succ_i y$ and $1 \geq p > q \geq 0$, then $px + ry \succ_i qx + ry$ (where $px + ry$ is the lottery which gives x, y, and D with probabilities p, r and $1 - p - r$ accordingly).*

WH (Weak Homogeneity): *Preferences $\succeq$ satisfy WH provided for any $x, y \in X$: $px \succeq y \Leftrightarrow \beta px \succeq \beta y$ for all $\beta \in (0, 1]$.*

C-CONVEXITY (Cardinal Convexity): *A bargaining problem, $(\succeq_1, \succeq_2)$, is C-Convex provided for any x, y in X, $x \neq y$, there is an outcome z in X such that for each i: $pb_i \sim_i x$ and $qb_i \sim_i y \Rightarrow \frac{1}{2}(p+q)b_i \preceq_i z$.*

[5] In a companion paper (Grant-Kajii (1994)) we provide a geometric characterization of a multi-agent extension of the RST ordinal Nash solution that enables us to define a class of preference relations that are compatible with the well known experimental violations of expected utility and for which the ordinal Nash solution is well defined.

The origin of the term "Cardinal" will become clear later. Note that C-Convexity is a joint hypothesis on preferences and the set X. It can be seen that X is connected if a problem is C-Convex; in particular, C-Convexity cannot be satisfied if X is finite.

The following is a restatement of RST's definition of the Ordinal Nash Outcome, and the reader is referred to RST for its interpretation. By convention, we refer to the agents by i and j.

DEFINITION 1: Let $x \in X$. We say that *agent i can appeal against* x, if there is an outcome $y \in X$ and $p \in [0, 1]$ such that $py \succ_i x$ and $px \prec_j y$. A Nash outcome is an outcome $x^* \in X$ against which neither agent can appeal.

We shall develop a cardinal characterization of the Nash outcome. First observe that the Nash outcome only involves risk preferences over elementary lotteries. So, let us consider risk preference relations which have an expected utility representation over the set of elementary lotteries.

DEFINITION 2: Preferences $\succeq_i$ are *disagreement linear (DL)* if there is a continuous function V_i from X to $\mathbb{R}_+$ such that for any $x, y \in X$:

$$px \succeq_i qy \Leftrightarrow V_i(x)p \geq V_i(y)q. \tag{1}$$

We call V_i a *DL representation.*

Since $0x \sim_i D$, we can set $V_i(D) = 0$ by convention. Obviously, expected utility preferences are DL. More generally, we have the following lemma.

LEMMA 1: *Suppose $\succeq_i$ satisfies DOM. Then preferences $\succeq_i$ satisfy WH if and only if they are DL. Moreover, DL representation is unique up to positive scalar multiplication.*

PROOF: Clearly, if preferences are DL, then they satisfy WH. Suppose WH holds. Recall that there is a best outcome b_i. For each $px \in \mathscr{L}_e$, we can find a unique number $\gamma(px) \in [0,1]$ such that $px \sim \gamma(px)b_i$, by DOM and (mixture) continuity. γ is continuous and $x \sim \gamma(1x)b_i$ by construction. Then by WH, $px \sim p\gamma(1x)b_i$, hence $\gamma(px) = p\gamma(1x)$ by the uniqueness of γ. Set $V_i(x) = \gamma(1x)$ for every $x \in X$, and it is straightforward to check that V_i is a DL representation. If V_i' is also a DL representation, then $V_i'(x) = V_i(x)V_i'(b_i)$ must hold for any x, since $p = V_i(x) \Rightarrow V_i'(b_i)p = V_i'(x)$. *Q.E.D.*

For a bargaining problem that admits DL representations for both agents' preferences, we have a simple *cardinal characterization* of the Nash outcome as follows:

LEMMA 2: *Suppose agents preferences are DL with DL representation V_i, $i = 1, 2$. Then $x^* \in X$ is a Nash outcome if and only if x^* is a solution to the following maximization problem*:

$$\max_{x \in X} V_1(x) \cdot V_2(x) \tag{2}$$

PROOF: Suppose x^* is a solution to (2). Suppose there is an outcome $y \in X$ and $p \in (0, 1]$ such that $py \succ_i x^*$ and $px^* \prec_j y$. That is, we have $V_i(y)p > V_i(x^*)$ and $V_j(x^*)p < V_j(y)$. Multiplying these two inequalities together we have $V_1(y)V_2(y)p > V_1(x^*)V_2(x^*)p$, a contradiction. So, x^* is a Nash outcome.

Conversely, if x^* is not a solution to (2), one can find y and $p \in (0, 1)$ such that $(V_2(y)/V_2(x^*)) > p > (V_1(x^*)/V_1(y))$. Then $V_1(y)p > V_1(x^*)$ and $V_2(x^*)p < V_2(y)$, which by definition implies that $py \succ_1 x^*$ and $px^* \prec_2 y$; that is, agent 1 can appeal against x^* with y. *Q.E.D.*

Although we agree with RST that the product of "utilities" is difficult to interpret, Lemma 2 does provide a useful operational method for determining the ordinal Nash solution and highlights how their ordinal definition is a natural preference based analog of the original "utility" based definition.

DEFINITION 3: Let $(\succeq_1, \succeq_2)$ be a bargaining problem where each $\succeq_i$ has a DL representation V_i. The *cardinal bargaining problem induced by* (V_1, V_2) is the set given by $S = \{(u_1, u_2): \exists x \in X, u_i \leq V_i(x) \text{ for both } i\}$.

An induced cardinal bargaining problem S is a comprehensive set by construction and it is closed since X is compact. To complete the analogy with Nash's original cardinal approach we also require the induced bargaining problem S to be convex.

LEMMA 3: *If C-Convexity holds for the bargaining problem* $(\succeq_1, \succeq_2)$ *and each* $\succeq_i$ *satisfies WH and DOM, then S is convex.*

PROOF: Suppose S is not convex. Then there are two outcomes x and y in X such that there is no $z \in X$ such that $(V_1(z), V_2(z))$ lies northeast of the line segment that connects $(V_1(x), V_2(x))$ and $(V_1(y), V_2(y))$. W.l.o.g., we can assume that for each i, V_i has the canonical form constructed as in the proof of Lemma 1. For each i, let p_i and q_i be probabilities defined by $p_i b_i \sim_i y$, i.e., $p_i = V_i(x)$ and $q_i = V_i(y)$ by construction. By C-Convexity, there exists $z \in X$ such that $z \succeq_i \frac{1}{2}(p_i + q_i)b_i$ for $i = 1, 2$. Since $V_i(z)b_i \sim_i z$, it follows from DOM that $V_i(z) \geq \frac{1}{2}(p_i + q_i) = \frac{1}{2}V_i(x) + \frac{1}{2}V_i(y)$ for both i, a contradiction. *Q.E.D.*

Our first result which improves upon RST's Proposition 2(a) (1992, p. 1180) follows as an immediate consequence of Lemmas 2 and 3, the simplifying assumption, and the compactness of X:

PROPOSITION 1: *Let* $(\succeq_1, \succeq_2)$ *be a bargaining problem where both* $\succeq_i$ *satisfy DOM and WH. Then a Nash outcome exists. If in addition* $(\succeq_1, \succeq_2)$ *is C-Convex, then the Nash outcome is unique.*

Our second result is a characterization of the *Nash solution*: Let $\mathscr{P}$ be a collection of preferences over $\mathscr{L}(X \cup \{D\})$ which satisfy the basic assumptions and DOM and WH. Let $\mathscr{B} \subset \mathscr{P} \times \mathscr{P}$ be the set of bargaining problems $(\succeq_1, \succeq_2)$ for which the Simplifying Assumption and C-Convexity hold. Let $\mathscr{B}'$ be any subset of $\mathscr{B}$. Then $\mathscr{B}'$ can be seen as a set of bargaining problems. An *ordinal bargaining solution* defined on $\mathscr{B}'$ is a function $F: \mathscr{B}' \to X$. The *Ordinal Nash Solution*, or simply the *Nash Solution*, is a function that assigns an ordinal Nash outcome to each bargaining problem in $\mathscr{P} \times \mathscr{P}$. We denote by $\mathscr{N}(\succeq_1, \succeq_2)$ the Nash outcome of $(\succeq_1, \succeq_2)$; then $\mathscr{N}$ is a well-defined ordinal bargaining solution by Proposition 1.

$(\succeq_1, \succeq_2)$ is said to be *symmetric* if there exists a function $\phi: X \cup \{D\} \to X \cup \{D\}$ which satisfies $y = \phi(x) \Leftrightarrow x = \phi(y)$ and $D = \phi(D)$ such that for all elementary lotteries px and qy, $px \succeq_i qy \Leftrightarrow p\phi(x) \succeq_j q\phi(y)$. ϕ is called a *symmetry function*.

Following RST, we shall consider the following preference based analogs of Nash's axioms for an ordinal bargaining solution F on $\mathscr{B}'$.

PAR: *For any* $(\succeq_1, \succeq_2) \in \mathscr{B}'$ *and* $x = F(\succeq_1, \succeq_2)$, *there is no* $y \in X$ *such that* $y \succ_i x$ *for both i.*

SYM: *If* $(\succeq_1, \succeq_2)$ *is a symmetric problem with a symmetry function* ϕ, *then* $F(\succeq_1, \succeq_2)$ *is a fixed point of the symmetry function* ϕ.

IIA: *Let* $x^* = F(\succeq_1, \succeq_2)$. *Suppose* $(\succeq'_1, \succeq'_2) \in \mathscr{B}'$, $i = 1, 2$, *satisfy*: (*i*) $\succeq'_i$ *agree with* $\succeq_i$ *on deterministic outcomes*; (*ii*) $x \succeq_i x^*$ *and* $px \sim_i x^*$ *implies* $x^* \succeq'_i px$; (*iii*) $x^* \succeq_i x$ *and* $x \sim_i qx^*$ *implies* $x \preceq'_i qx^*$. *Then* $x^* = F(\succeq'_1, \succeq'_2)$.

The following extends RST's Proposition 2(c) (1992, p. 1180):

PROPOSITION 2: *Let* $\mathscr{B}' \subset \mathscr{B}$ *and suppose* $\mathscr{B}'$ *contains all expected utility preferences, and at least one symmetric problem.*[6] *Then the ordinal Nash solution is the unique bargaining solution defined on* $\mathscr{B}'$ *which satisfies PAR, SYM, and IIA.*

It is straightforward to check that the Nash Solution $\mathscr{N}$ satisfies PAR, SYM, and IIA on any $\mathscr{B}'$. Uniqueness follows from Lemma 2 by a straightforward application of the well-known, classical theorem of Nash (1950) (see Appendix).

3. COMPARISIONS WITH RST'S RESULTS AND SOME REMARKS

For risk preferences, RST assumed, in addition to DOM, the following:

H (Homogeneity): *If* $x \succeq_i L$, *then* $\alpha x \succeq_i \alpha L$ *and if* $x \sim_i L$, *then* $\alpha x \sim_i aL$.

Q (Quasi-Concavity): *If* $L' \succ_i L$, *then for any* $\alpha \in [0, 1]$, $\alpha L' + (1-\alpha)L \succ_i L$.

CCE (Conditional Certainty Equivalence): *If* $x \sim_i L'$, *then* $\alpha x + (1-\alpha)z \succeq_i \alpha L' + (1-\alpha)z$ *for any* z *such that* $z \succeq_i y$ *in the support of* L'.

RST also show that Q and CCE can be replaced with the following pair of assumptions:

WQ: *If* $L' \succ_i x$, *then for any* $\alpha \in [0, 1]$, $\alpha L' + (1-\alpha)x \succ_i x$.

CCE*: *If* $x \succeq_i L'$, *then* $\alpha x + (1-\alpha)L \succeq_i \alpha L' + (1-\alpha)L$ *for any lottery* L.

Instead of C-Convexity, RST required the following alternative notion of convexity to hold for bargaining problems.

CONVEXITY: *A bargaining problem* $(\succeq_1, \succeq_2)$ *is convex if for any* x, y *in* X, *there exists an outcome* z *in* X *such that* $z \succeq_i \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$ *for both* i.

H, as RST readily admits, is a very strong axiom that states that risk preferences are homothetic with respect to mixtures with the disagreement outcome D. That is, indifference surfaces in the probability simplex are homothetic with respect to the D vertex. Our assumption WH is strictly weaker than H, since it only restricts preferences of lotteries that lie along the "edges" connecting the D vertex to any other vertex.

Q (quasi-concavity) and its weaker version WQ are not necessarily any more plausible than quasiconvexity. In fact some authors have argued that quasiconcave preferences are vulnerable to Dutch books.[7] Moreover, for one well-known class of non-expected utility

[6] The existence of a symmetric problem will be trivially satisfied if X is a nonempty, compact and convex subject of $\mathbb{R}^n_+$ and $\mathscr{B}'$ contains all expected utility preferences.

[7] Green (1987), for instance.

risk preferences, rank dependent expected utility, quasiconcavity is incompatible with aversion to mean preserving spreads.[8] RST defend Q by noting that they can allow preferences to be both quasi-concave and quasi-convex (that is, the indifference surfaces in the simplex are "planar"). But notice that, in conjunction with H, this implies that the preferences restricted to lotteries over any triple of outcomes that includes the disagreement outcome must be representable by an expected utility functional. That is, the indifference curves in the simplex are linear *and* parallel. We simply note that a wide variety of the experimentally observed violations of expected utility theory have involved choices over three outcomes, where the worst outcome might quite reasonably be interpreted as the disagreement outcome in a bargaining context.

CCE is defended by RST on the grounds that it is satisfied whenever preferences satisfy "fanning out" (Machina's (1982) hypothesis II) but again it seems restrictive to rule out on a priori reasoning "fanning in," particularly since it too has received support in reported experimental evidence.[9]

To sum up, Q, WQ, CCE, and CCE* tend to exclude interesting and popular classes of non-expected utility preferences despite RST's motivation. On the other hand, none of these uncomfortable assumptions is required for any of our results.

It is not difficult to see that in general, C-Convexity neither implies nor is implied by Convexity. However, under H and Q, a Convex bargaining problem is C-Convex. This follows immediately from the property that $pb_i \sim_i x$ and $qb_i \sim_i y$ implies $\frac{1}{2}\delta_x + \frac{1}{2}\delta_y \succeq_i \frac{1}{2}(p+q)b_i$, under H and Q. To see this, say $x \preceq_i' y$. Then by H, $x \sim_i ry$ where $r \equiv p/q \leq 1$, and so $(r/(1+r))\delta_x + (1/(1+r))(r\delta_y + (1-r)D) = s[\frac{1}{2}\delta_x + \frac{1}{2}\delta_y] \succeq_i ry$ where $s \equiv (2r/(r+1))$ by Q. On the other hand, $ry \sim_i (rq)b_i = s((p+q)/2)b_i$ by H, so the property holds.

In conclusion, our assumptions are implied by RST's assumptions. To show that Propositions 1 and 2 are strict extensions of their counterparts in RST, we shall give an example of a bargaining problem with RDEU preferences, which satisfies all of our assumptions but none of the four RST assumptions Q, WQ, CCE, and CCE*.

EXAMPLE 1: Let $X = \{(x_1, x_2) \in \mathbb{R}_+^2 | x_1 + x_2 \leq 1\}$ and $D = (0, 0)$. Agent i's preferences only depend on his share, x_i, of the "pie" over which the agents are bargaining. Consider preferences of agent i represented by the following RDEU functional:

$$U_i^*(L) = \int_{z=0}^{1} (G_i(z))^{\alpha_i}\, dz \tag{3}$$

where G_i denotes the decumulative distribution function of i's share of the pie induced by L and $\alpha_i > 1$ for $i = 1, 2$.[10]

It is straightforward to see that for this bargaining problem both preference relations satisfy H (and hence WH) and $V_i(x) = x_i^{1/\alpha_i}$ is a DL representation of $\succeq_i$. Aversion to risk (in the Rothschild-Stiglitz sense) implies and is implied by $\alpha_i > 1$ for $i = 1, 2$.[11] So, the problem is Convex. It is also C-Convex: to see this, note that $U_i^*(pb_i) = p^{\alpha_i}$ and that $p^{\alpha_i} = x_i$ and $q^{\alpha_i} = y_i$ implies $z \equiv \frac{1}{2}x_i + \frac{1}{2}y_i \geq [\frac{1}{2}p + \frac{1}{2}q]^{\alpha_i} = U_i^*([\frac{1}{2}p + \frac{1}{2}q]b_i)$. From (3), it follows that $\alpha_i > 1$ implies that for any L, L' with $U_i^*(L) = U_i^*(L')$, $U_i^*(pL + (1-p)L') \leq pU_i^*(L) + (1-p)U_i^*(L')$; that is, U_i^* is quasi-*convex* in probabilities. From this, it is easy to see that Q is not satisfied, and that CCE also fails to hold.

[8] See Theorem 1 of Chew et al. (1987, p. 374).

[9] See, for instance, Camerer (1989), Starmer (1992).

[10] This functional is an example of the "dual model" axiomatized by Yaari (1987) where the decumulative distribution function for a random variable X is defined as

$$G_i(x) = \Pr(X > z).$$

[11] See Chew et al. (1987, Theorem 1, p. 374).

Applying Lemma 2, the ordinal Nash outcome corresponds to the split of the dollar that solves

$$\max_{z \in [0,1]} z^{1/\alpha_i} \cdot (1-z)^{1/\alpha_2}.$$

Let us point out that the risk preferences considered in Example 1 do not accord a special status to the disagreement outcome. Rather the preference between two lotteries is maintained when they are both mixed with *any* outcome that is worse than any outcome in the supports of those two lotteries. Hence for this type of preference relation, our results could be readily generalized to bargaining problems where the set of alternatives and the disagreement outcome varied. This appears to be an attractive feature since the disagreement outcome (outside option) may well be endogenously determined in some applications.

Although WH is weaker than H, it is nevertheless a substitution axiom similar in nature to the independence axiom. In particular, WH is not consistent with the *common ratio effect*. Let x and y be two outcomes corresponding to \$3000 and \$4000 respectively, and let D be the outcome \$0.

PROBLEM 1: Choose either x or $0.8y$.

PROBLEM 2: Choose either $0.25x$ or $0.2y$.

Many studies have shown a systematic tendency for subjects when faced with such problems to express a preference for x over $0.8y$ in the first problem and for $0.2y$ over $0.25x$ in the second problem constituting a direct violation of WH.[12]

Without WH, however, the existence of an ordinal Nash solution is more delicate. We illustrate this point in the following example where the risk preferences can accommodate violations such as the common ratio effect.

EXAMPLE 2: Consider the setup as in Example 1, except that preferences are represented by

$$U_i^*(L) = \int_{z=0}^{1} [1-(1-G_i(z))^{\alpha_i}]\,dz \tag{4}$$

where $\alpha_i > 0$. The problem can be shown to be C-Convex if $\alpha_i \leq 1$ for each $i = 1, 2$. But note that $U_i(px) = x_i(1-(1-p)^{\alpha_i})$ for elementary lotteries, indicating that WH is violated. Moreover, if $\alpha_i < 1$, then it is not difficult to see that the preferences are consistent with the common ratio effect.[13]

However, except for the case where $\alpha_1 = \alpha_2$, the ordinal Nash solution does not exist for this problem. Obviously, regardless of the particular values of α_1 and α_2, the split (0, 1) and (1, 0) is not a Nash outcome, so let $x \in (0, 1)$. If $\alpha_1 > \alpha_2$, it is straightforward to verify that the function $y[1-(1-p)^{\alpha_1}]$ is increasing with respect to y at $y = x$ subject to the constraint $(1-y) = (1-x)[1-(1-p)^{\alpha_2}]$, so player 1 is able to successfully appeal against any split $(x, 1-x)$ with $x < 1$. Intuitively, although both players are risk averse, if $\alpha_1 > \alpha_2$, then player 1 is relatively less risk averse for small *probabiity* risks around any x and is thus able to obtain a concession from player 2 indefinitely. Similarly, if $\alpha_2 > \alpha_1$, then player 2 can successfully appeal against any split $(x, 1-x)$. Only in the case where $\alpha_1 = \alpha_2$ will there exist an ordinal Nash solution, which corresponds to the split (1/2, 1/2) that maximizes the expression $x^*(1-x^*)$.

12 Kahneman and Tversky (1979), for instance.

13 Grant-Kajii (1993) discuss this type of preferences and show that concavity of u_i and $\alpha_i \leq 1$ are in fact necessary and sufficient for an individual to be risk averse.

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APPENDIX

PROOF OF PROPOSITION 2: Let us say $(u_1^*, u_2^*) \in S$ is a *cardinal Nash outcome* of S if $u_1^* \cdot u_2^* \geq u_1 \cdot u_2$ for any $(u_1, u_2) \in S$. Let $\nu(S)$ be the cardinal Nash outcome of S.

NASH'S THEOREM: *Let $\mathscr{S}$ be a collection of convex, comprehensive subsets of R_+^2 such that $S_1 \equiv \{s \in R_+^2 : s_1 + s_2 \leq 1\} \in \mathscr{S}$ and $\lambda S_1 \in \mathscr{S}$ for any $\lambda \in \Re_{++}^2$.*[14] *Let $f: \mathscr{S} \to 2^{\Re^2} \setminus \varnothing$ be a set valued function such that $f(S) \subset S$ for all $S \in \mathscr{S}$. Suppose the following properties hold:*[15] *(a) $f(\lambda S) = \lambda f(S)$ for any $\lambda \in R_{++}^2$; (b) $f(S_1) = \{(\frac{1}{2}, \frac{1}{2})\}$; (c) if $f(S_1) \subset S$ and $S \leq S_1$, then $f(S) = f(S_1)$. Then $f(S) = \{\nu(S)\}$ for all $S \in \mathscr{S}$.*

Let $\mathscr{S}_I$ be the set of all cardinal bargaining problems induced by some $(\succeq_1, \succeq_2) \in \mathscr{B}'$. Since the DL representation is unique up to positive scalar multiplication, the set of all cardinal bargaining problems corresponding to a bargaining problem $(\succeq_1, \succeq_2)$ has the following simple form: let (V_1, V_2) be any DL representations of $(\succeq_1, \succeq_2)$ and let S be the induced cardinal problem. Then the set of all cardinal bargaining problems corresponding to $(\succeq_1, \succeq_2)$ is $\{aS : a \in R_{++}^2\}$.

Pick a symmetric problem, $(\succeq_1, \succeq_2) \in \mathscr{B}'$, with associated symmetry function ϕ. Since $(\succeq_1, \succeq_2)$ is C-Convex, w.l.o.g., let (V_1, V_2) be its DL representation with range$(V_1) = [0, 1]$, and let S be the induced cardinal problem. For each $x \in X$, set $U_i(x) = \alpha(x) V_i(x)$, where $\alpha(x) = \inf\{\alpha > 0: ((1/\alpha)V_1(x)), ((1/\alpha)V_2(x)) \in S\}$. Let $(\succeq_1', \succeq_2')$ be the bargaining problem with EU preferences whose VNM utility functions are (U_1, U_2). By construction, $(\succeq_1', \succeq_2')$ is also a symmetric problem with symmetry function ϕ and it induces S_1. This in particular shows that $\mathscr{S}_I$ satisfies the requirements in Nash's Theorem.

Let $\mathscr{B}' \subset \mathscr{B}$ and let F be any ordinal bargaining problem defined on $\mathscr{B}'$ that satisfies PAR, SYM, and IIA. Define a set valued function $f: \mathscr{S} \to 2^{\Re^2} \setminus \varnothing$ by the rule:

$$f(S) = \{(V_1(x), V_2(x)) : \exists (\succeq_1, \succeq_2) \text{ with } x = F(\succeq_1, \succeq_2) \text{ that induces } S \text{ with } V_i\}.$$

Note that it is possible that two ordinal bargaining problems, $(\succeq_1, \succeq_2)$ with DL representation (V_1, V_2) and $(\succeq_1', \succeq_2')$ with (V_1', V_2'), induce the same cardinal bargaining problem S. But it is not necessarily true that $(V_1(x), V_2(x)) = (V_1'(x'), V_2'(x'))$, where $x = F(\succeq_1, \succeq_2)$ and $x' = F(\succeq_1', \succeq_2')$. So in general $f(S)$ is a nontrivial, set valued function even if F is singleton valued. But if $f(S) = \{v(S)\}$, then by Lemma 2, $F(\succeq_1, \succeq_2) = \mathscr{N}(\succeq_1, \succeq_2)$ must follow whenever $(\succeq_1, \succeq_2)$ induces S.

Therefore, Proposition 2 will be established if we can show that f satisfies (a), (b), and (c) in Nash's Theorem. By construction and by the essential uniqueness of DL representation, (a) is clearly satisfied. To see that (b) is satisfied, consider the bargaining problem $(\succeq_1', \succeq_2')$ constructed above that has EU representations (U_1, U_2) and induces S_1. Let $x^* = F((\succeq_1', \succeq_2'))$. By SYM, $x^* = \phi(x^*)$ which implies that $U_1(x^*) = U_2(x^*)$. By PAR, $U_1(x^*) + U_2(x^*) = 1$. So, $f(S_1) = \{(\frac{1}{2}, \frac{1}{2})\}$.

To see that (c) is satisfied, pick any problem $(\succeq_1, \succeq_2)$ that induces a cardinal bargaining problem $S \in \mathscr{S}_I$ such that $(\frac{1}{2}, \frac{1}{2}) \in S \leq S_1$. Let V_1 be the corresponding DL representations and x^* be a corresponding outcome to $(\frac{1}{2}, \frac{1}{2})$ (which is uniquely determined by the simplifying assumptions). Since $f(S_1) = \{(\frac{1}{2}, \frac{1}{2})\}$, it is sufficient to show that $F((\succeq_1, \succeq_2)) = x^*$ by IIA. For each $s \in S$, let $\tau[s] = \sup\{t > 0 : ts \leq S_1\}$. Let $(u_1(x), u_2(x)) = \tau[(V_1(x), V_2(x))](V_1(x), V_2(x))$. Let $\succeq_i'$ be EU preferences defined by u_i. By the simplifying assumption, $\succeq_i$ and $\succeq_i'$ induce the same ordering over X.

[14] For any set A, $B \in R^2$ and a vector $r \in R^2$, "$A \leq B$" means "$\forall a \in A$, $\exists b \in B$, $a \leq b$" and $rA = \{r_1 a_1, r_2 a_2) : (a_1, a_2) \in A\}$.

[15] (a) is *scale independence*, (b) is *symmetry*, and (c) is *independence from irrelevant alternatives*.

Since $S \leq S_1$, $u_i(x) \geq V_i(x)$ for all x, and $u_i(x) = V_i(x) = \frac{1}{2}$. Since $(\succeq'_1, \succeq'_2)$ induces S_1, we have $F((\succeq'_1, \succeq'_2)) = x^*$. Then conditions (ii) and (iii) in the definition of IIA are satisfied at x^* for $(\succeq_1, \succeq_2)$, $(\succeq'_1, \succeq'_2)$. To see this, recall that $px \sim_i x^*$ if and only if $pV_i(x) = \frac{1}{2}$ by construction. So, for each i, (ii) is equivalent to saying that $u_i(x) \geq V_i(x)$ for all $x \succeq_i x^*$, $x \in X$. Similarly, (iii) is equivalent to $u_i(x) \geq V_i(x)$ for all $x \succeq_i x^*$, $x \in X$. So, by IIA, we obtain $x^* = F((\succeq'_1, \succeq'_2)) = F((\succeq_1, \succeq_2))$. Consequently, $F(S) = f(S_1)$, as we desired. *Q.E.D.*

REFERENCES

CAMERER, C. (1989): "An Experimental Test of Several Generalized Utility Theories," *Journal of Risk and Uncertainty*, 2, 61–104.

CHEW, S. H. (1989): "Axiomatic Utility Theories with the Betweenness Property," *Annals of Operations Research*, 19, 273–298.

CHEW, S. H., E. KARNI, AND Z. SAFRA (1987): "Risk Aversion in the Theory of Expected Utility with Rank Dependent Probabilities," *Journal of Economic Theory*, 42, 304–318.

GRANT, S., AND A. KAJII (1993): "AUSI EU Preferences: A Rank Dependent Expected Utility Theory with Disappointment Aversion," CARESS Working Paper #93-04, University of Pennsylvania.

——— (1994): "Bargaining, Boldness and Nash Outcomes with Non-expected Utility," ANU Working Paper #264.

GREEN, J. (1987): "'Making Book Against Oneself': The Independence Axiom and Nonlinear Utility," *Quarterly Journal of Economics*, 102, 785–796.

KAHNEMAN, D., AND A. TVERSKY (1979): "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica*, 47, 263–291.

MACHINA, M. (1982): "'Expected Utility' Analysis without the Independence Axiom," *Econometrica*, 50, 277–323.

NASH, J. (1950): "The Bargaining Problem," *Econometrica*, 28, 155–162.

QUIGGIN, J. (1992): *Generalized Expected Utility Theory: The Rank-Dependent Model.* Dordrecht: Kluwer Academic Publishers.

RUBINSTEIN, A., Z. SAFRA, AND W. THOMSON (1992): "On the Interpretation of the Nash Bargaining Solution and its Extension to Non-expected Utility Preferences," *Econometrica*, 60, 1171–1186.

STARMER, C. (1992): "Testing New Theories of Choice under Uncertainty Using the Common Consequence Effect," *Review of Economic Studies*, 59, 813–830.

VALENCIANO, V., AND J. M. ZARZUELO (1994): "On the Interpretation of Nonsymmetric Bargaining Solutions and their Extension to Non-expected Utility Preferences," *Games and Economic Behavior*, 7, 461–472.

YAARI, M. (1987): "The Dual Theory of Choice under Risk," *Econometrica*, 55, 95–115.