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On the equivalence of preferences

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Abstract

Preference relations that agree on the ranking of elements of the space on which they are defined necessarily agree on all the conditional rankings of pairs of components of these elements. In this note we show the converse statement is also true.

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1. Introduction

Preference relations that agree on the ranking of elements of the space on which they are defined necessarily agree on all the conditional (on the remaining components) rankings of pairs of components of these elements. In this note we show the converse statement is also true. More precisely, agreement on all the conditional rankings of pairs of components implies an agreement of the preference relations on the entire space. For example, if two preference relations agree on the rankings of intertemporal consumption in consecutive pairs of adjacent periods, holding the rest of the consumption stream constant, then they agree on the ranking of the entire set of consumption streams. The same claim applies to partial agreement (i.e. agreement on subsets of the space on which the preference relations are defined). This result is used in Grant and Karni (2000) in the context of the theory of decision making under uncertainty.

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2. The model

Let $(X_i)_{i\in\mathbb{Z}}$, be a countable collection of topological spaces. Set $\mathbf{X}$ to be the Cartesian product $\prod_{i\in\mathbb{Z}}X_i$. Write $\mathbf{X}_{-i}$ for $\prod_{k\neq i}X_k$ and similarly, write $\mathbf{X}_{-(i,j)}$ for $\prod_{k\neq i,j}X_k$. For each $\mathbf{x}\in\mathbf{X}$,

$$\mathbf{x}_{-i}=(\ldots,x_{i-1},x_{i+1},\ldots,)\in\mathbf{X}_{-i}$$

and

$$\mathbf{x}_{-(i,j)}=(\mathbf{x}_{-i})_{-j}=(\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1},x_{j+1},\ldots)\in\mathbf{X}_{-(i,j)}.$$

Furthermore, for each $\mathbf{x}\in\mathbf{X}$, $y\in X_i$ and $z\in X_j$,

$$\begin{aligned}(y,\mathbf{x}_{-i}) &= (\ldots,x_{i-1},y,x_{i+1},\ldots)\in\mathbf{X}\\ ((y,z),\mathbf{x}_{-(i,j)}) &= (\ldots,x_{i-1},y,x_{i+1},\ldots,x_{j-1},z,x_{j+1},\ldots)\in\mathbf{X}\end{aligned}$$

Let $\mathcal{U}$ be the set of continuous functions (in the product topology) of the form $u{:}\mathbf{X}\to\mathbb{R}$, that satisfy the following property.

Property 1. (Co-ordinate-wise monotonicity and solvability) A function $u{:}\mathbf{X}\to\mathbb{R}$ is co-ordinate-wise monotonic and solvable (MS) if for each $i\in\mathbb{Z}$, there exists a function $f_i{:}\mathbb{R}\to X_i$, such that for every $\mathbf{x}\in\mathbf{X}$**:**

1. there exists $\alpha\in\mathbb{R}$, such that $u(f_i(\alpha),\ \mathbf{x}_{-i})=u(\mathbf{x})$;
2. $u(f_i(\beta),\ \mathbf{x}_{-i})>u(\mathbf{x})\Rightarrow\beta>\alpha$;
3. if $\{y\in X_i{:}\ u(y,\ \mathbf{x}_{-i})>u(\mathbf{x})\}\neq\emptyset$, then $\beta>\alpha\Rightarrow u(f_i(\beta),\ \mathbf{x}_{-i})>u(f_i(\alpha),\ \mathbf{x}_{-i})$.

We shall refer to a collection of such functions $\{f_i\}_{i\in\mathbb{Z}}$ as a set of functions that is *congruent* with the MS property for u.

The MS property implies a co-ordinate independence property. In particular, we can view the range of f_i, as a subset of X_i for which there are no 'conditional (strict) preference reversals' of the following form:

$$u(f_i(\alpha),\mathbf{x}_{-i})>u(f_i(\beta),\mathbf{x}_{-i})\text{ and }u(f_i(\alpha),\mathbf{y}_{-i})<u(f_i(\beta),\mathbf{y}_{-i}).$$

- An immediate implication of the MS property, is the following. For a given $u\in\mathcal{U}$ and a collection of functions, $\{f_i\}_{i\in\mathbb{Z}}$, congruent with the MS property for u, for any $\mathbf{x}$ that is not maximal, there exists a unique $\lambda\in(0,1)$, such that $u(\mathbf{x})=u((\ldots,\ f_i(\lambda),\ \ldots,))$. Moreover, for any $\gamma>\lambda$, we have $u((\ldots,\ f_i(\gamma),\ \ldots,))>u(\mathbf{x})$.
- If there exists either a maximal element $\overline{\mathbf{x}}$ and/or a minimal element $\underline{\mathbf{x}}$, we shall select a 'normalized' set of functions, $\{f_i\}_{i\in\mathbb{Z}}$, congruent with the MS property for u, for which

$$u(\underline{\mathbf{x}})=u((\ldots,\ f_i(0),\ \ldots,))\text{ and }u(\overline{\mathbf{x}})=u((\ldots,\ f_i(1),\ \ldots,)).$$

Similarly, if for any co-ordinate, there exists a maximal (resp. minimal) element $f_i(\lambda)\in X_i$ (resp. $f_i(\gamma)\in X_i$) in the sense that

$$
\begin{aligned}
&u(f_i(\lambda), \mathbf{x}_{-i}) &&\geq && u(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{X} \\
&(\text{resp. } u(f_i(\gamma), \mathbf{x}_{-i}) &&\leq && u(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{X})
\end{aligned}
$$

then we shall select a normalized f_i for which

$$
\begin{aligned}
&u(f_i(1), \mathbf{x}_{-i}) &&\geq && u(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{X} \\
&(\text{resp. } u(f_i(0), \mathbf{x}_{-i}) &&\leq && u(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{X})
\end{aligned}
$$

Definition 1. For a given $u \in \mathcal{U}$ and a collection of functions, $\{f_i\}_{i \in \mathbb{Z}}$, congruent with the MS property for u, the coordinate i is *essential* if f_i is not a constant function. Any coordinate that is not essential shall be dubbed *inessential*.

3. Main result

Let G be a reflexive and symmetric binary relation over $\mathbb{Z}$. That is, for all $i, j \in \mathbb{Z}$ (i) $(i, i) \in G$ and (ii) $(i, j) \in G$ implies $(j, i) \in G$. We can view G as a (undirected) graph, with vertices $\mathbb{Z}$, and an edge connecting vertices i and j, if $(i, j) \in G$. For the 'graph' to be connected, we require that for any pair of vertices $i, j \in \mathbb{Z}$, there exists a (finite) sequence $\{k_n\}_{n=1}^N$, such that $k_1 = i$, $k_N = j$ and $(k_n, k_{n+1}) \in G$, for all $n = 1, \ldots, N-1$. That is, for the graph to be connected, we require that there exists a finite path from any vertex to any other vertex. It is straightforward to see that connectedness is equivalent to the transitive closure of G being a complete binary relation which in conjunction with reflexivity and symmetry implies it is equal to $\mathbb{Z} \times \mathbb{Z}$. Similarly, for G restricted to a subset of coordinates, $I \subset \mathbb{Z}$, the restricted relation is complete if its transitive closure equals $I \times I$.

Definition 2. A pair of functions u and $\hat{u}$ from $\mathcal{U}$, conditionally agree on $X_i \times X_j$, with $i \neq j$, if for any $\mathbf{x}_{-(i,j)} \in \mathbf{X}_{-(i,j)}$, $u((\cdot, \cdot), \mathbf{x}_{-(i,j)})$ and $\hat{u}((\cdot, \cdot), \mathbf{x}_{-(i,j)})$ induce the same (weak) ordering on $X_i \times X_j$.

Theorem 1. *Fix $u, \hat{u} \in \mathcal{U}$. Let G be the reflexive and symmetric binary relation corresponding to: u and $\hat{u}$ conditionally agree on $X_i \times X_j$ if and only if $(i, j) \in G$. The following are equivalent.*

(*i*) *The transitive closure of G restricted to the set of essential coordinates is complete.*
(*ii*) *u and $\hat{u}$ induce the same weak ordering on* $\mathbf{X}$.

Proof. (ii)⇒(i) is immediate.

(i)⇒(ii) (By induction). Fix $\mathbf{x} \in \mathbf{X}$ and for ease of exposition (and without essential loss of generality) assume that $\mathbf{x}$ is not maximal with respect to any of the essential coordinates. Let I be the set of essential coordinates. If $I = \emptyset$, then both u and $\hat{u}$ are constant functions and hence the result holds trivially. If $|I| = 1$ then the conclusion follows, by definition, from Property 1. So take $|I| > 1$. Since the transitive closure of G restricted to I is non-empty, it follows that G restricted to I is non-empty. So fix some pair of essential coordinates $(i, j) \in G$. By definition u and $\hat{u}$ conditionally agree on $X_i \times X_j$. If $|I| = 2$, these are the only essential coordinates and hence the result holds. If $|I| \geq 3$, since the transitive closure of G restricted to I is $I \times I$, there exists an essential coordinate $k \neq i, j$, for which either $(i, k) \in G$ or $(j, k) \in G$. Without loss of generality, assume $(j, k) \in G$.

Since $(i, j) \in G$ and u and $\hat{u}$ exhibit the MS property, it follows that there exists a unique $\beta^1 \in \mathbb{R}$, such that, for all x_i', x_j' and any given $\mathbf{x}_{-(i,j)}$

$$u(x_i', x_j', \mathbf{x}_{-(i,j)}) = u\big((f_i(\beta^1), f_j(\beta^1)), \mathbf{x}_{-(i,j)}\big) \Leftrightarrow \hat{u}(x_i', x_j', \mathbf{x}_{-(i,j)}) = \hat{u}\big((f_i(\beta^1), f_j(\beta^1)), \mathbf{x}_{-(i,j)}\big).$$

Since $(j, k) \in G$ and u and $\hat{u}$ exhibit the MS property, it follows there exists a unique $\beta^2 \in \mathbb{R}$, such that, for all x_j', x_k',

$$\begin{aligned} u(x_j', x_k', (x_i', \mathbf{x}_{-i})_{-(j,k)}) &= u\big((f_j(\beta^2), f_k(\beta^2)), (f_i(\beta^1), \mathbf{x}_{-i})_{-(j,k)}\big) \Leftrightarrow \\ \hat{u}(x_j', x_k', (x_i', \mathbf{x}_{-i})_{-(j,k)}) &= \hat{u}\big((f_j(\beta^2), f_k(\beta^2)), (f_i(\beta^1), \mathbf{x}_{-i})_{-(j,k)}\big). \end{aligned}$$

If $\beta^1 = \beta^2$ we can proceed to the next stage. Otherwise, without loss of generality, assume $\beta^2 > \beta^1$.

Since $(i, j) \in G$ and u and $\hat{u}$ exhibit the properties monotonicity and solvability, it follows there exists a unique $\beta^3 \in \mathbb{R}$, such that,

$$\begin{aligned} u(x_i', x_j', (x_i', \mathbf{x}_{-k})_{-(i,j)}) &= u\big((f_i(\beta^3), f_j(\beta^3)), (f_k(\beta^2), \mathbf{x}_{-k})_{-(i,j)}\big) \Leftrightarrow \\ \hat{u}(x_i', x_j', (x_i', \mathbf{x}_{-k})_{-(i,j)}) &= \hat{u}\big((f_i(\beta^3), f_j(\beta^3)), (f_k(\beta^2), \mathbf{x}_{-k})_{-(i,j)}\big) \end{aligned}$$

By the MS property $\beta^3 \in (\beta^1, \beta^2)$. To see this, notice that if $\beta^3 \geq \beta^2$ then

$$u\big((f_i(\beta^3), f_j(\beta^3)), (f_k(\beta^2), \mathbf{x}_{-k})_{-(i,j)}\big) > u\big((f_j(\beta^2), f_k(\beta^2)), (f_i(\beta^1), \mathbf{x}_{-i})_{-(j,k)}\big),$$

which is a contradiction. Similarly, if $\beta^3 \leq \beta^1$ then

$$u\big((f_j(\beta^2), f_k(\beta^2)), (f_i(\beta^1), \mathbf{x}_{-i})_{-(j,k)}\big) > u\big((f_i(\beta^3), f_j(\beta^3)), (f_k(\beta^2), \mathbf{x}_{-k})_{-(i,j)}\big).$$

Since $(j, k) \in G$ and u and $\hat{u}$ exhibit the MS property, it follows there exists a unique $\beta^4 \in \mathbb{R}$, such that,

$$\begin{aligned} u(x_j', x_k', (x_i', \mathbf{x}_{-i})_{-(j,k)}) &= u\big((f_j(\beta^4), f_k(\beta^4)), (f_i(\beta^3), \mathbf{x}_{-i})_{-(j,k)}\big) \Leftrightarrow \\ \hat{u}(x_j', x_k', (x_i', \mathbf{x}_{-i})_{-(j,k)}) &= \hat{u}\big((f_j(\beta^4), f_k(\beta^4)), (f_i(\beta^3), \mathbf{x}_{-i})_{-(j,k)}\big) \end{aligned}$$

By the MS property $\beta^4 \in (\beta^3, \beta^2)$. To see this, notice that if $\beta^4 \geq \beta^2$ then

$$u\big((f_j(\beta^4), f_k(\beta^4)), (f_i(\beta^3), \mathbf{x}_{-i})_{-(j,k)}\big) > u\big((f_i(\beta^3), f_j(\beta^3)), (f_k(\beta^2), \mathbf{x}_{-k})_{-(i,j)}\big)$$

which is a contradiction. And, similarly, if $\beta^4 \leq \beta^3$, then

$$u\big((f_i(\beta^3), f_j(\beta^3)), (f_k(\beta^2), \mathbf{x}_{-k})_{-(i,j)}\big) > u\big((f_j(\beta^4), f_k(\beta^4)), (f_i(\beta^3), \mathbf{x}_{-i})_{-(j,k)}\big).$$

Reiterating the same operation we obtain two sequences: $\beta_1 < \beta_3 < \ldots < \beta_{2i+1} < \ldots$ and $\beta_0 > \beta_2 > \beta_4 > \ldots > \beta_{2i} > \ldots$. Both sequences are bounded above by β_2 and below by β_1, and so the first sequence has a least upper bound, $\underline{b}$, and the second sequence has a greatest lower bound, $\bar{b}$. Furthermore by the MS property there exists a unique $\beta \in \mathbb{R}$, such that

$$u(x'_j, x'_k, (x'_i, \mathbf{x}_{-i})_{-(j,k)}) \quad = \quad u((f_j(\beta), f_k(\beta)), (f_i(\beta), \mathbf{x}_{-i})_{-(j,k)})$$

and

$$\hat{u}(x'_j, x'_k, (x'_i, \mathbf{x}_{-i})_{-(j,k)}) \quad = \quad \hat{u}((f_j(\beta), f_k(\beta)), (f_i(\beta), \mathbf{x}_{-i})_{-(j,k)}).$$

Thus, by the MS property it follows that $\underline{b} = \beta = \overline{b}$.

Assume that this holds for any collection of n essential coordinates for which the transitive closure of G restricted to n is complete. In particular, suppose that the first $n+1$ coordinates are all essential and that the transitive closure of G restricted to the first $n-1$ coordinates is complete and that $(n-1, n) \in G$ and $(n, n+1) \in G$. By the induction assumption there exists a unique $\beta^1 \in \mathbb{R}$ such that, for all $(x'_1, \ldots, x'_n)$ and any given $(x_{n+1}, \ldots)$

$$u(x'_1, \ldots, x'_n, x_{n+1}, \ldots) = u\left(\begin{bmatrix} f_i(\beta^1) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^1) & \text{on} & n \\ x_{n+1} & \text{on} & n+1 \\ x_j & \text{on} & j \not\in \{1, \ldots, n+1\} \end{bmatrix}\right) \Leftrightarrow$$

$$\hat{u}(x'_1, \ldots, x'_n, x_{n+1}, \ldots) = \hat{u}\left(\begin{bmatrix} f_i(\beta^1) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^1) & \text{on} & n \\ x_{n+1} & \text{on} & n+1 \\ x_j & \text{on} & j \not\in \{1, \ldots, n+1\} \end{bmatrix}\right).$$

Since $(n, n+1) \in G$ and u and $\hat{u}$ exhibit the MS property, it follows there exists a unique $\beta^2 \in \mathbb{R}$, such that for all $(x'_1, \ldots, x'_{n+1})$ and any $(x_{n+2}, \ldots)$

$$\begin{aligned} &u(x'_1, \ldots, x'_{n+1}, x_{n+2}, \ldots) \\ &= u\left(\begin{bmatrix} f_i(\beta^1) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^2) & \text{on} & n \\ f_{n+1}(\beta^2) & \text{on} & n+1 \\ x_j & \text{on} & j \not\in \{1, \ldots, n+1\} \end{bmatrix}\right) \Leftrightarrow \end{aligned}$$

$$\hat{u}(x'_1, \ldots, x'_{n+1}, x_{n+2}, \ldots) = \hat{u}\left[\begin{bmatrix} f_i(\beta^1) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^2) & \text{on} & n \\ f_{n+1}(\beta^2) & \text{on} & n+1 \\ x_j & \text{on} & j \not\in \{1, \ldots, n+1\} \end{bmatrix}\right].$$

If $\beta^1 = \beta^2$ then we are done. And we can proceed to the next stage. Otherwise, either $\beta^1 > \beta^2$ or $\beta^2 > \beta^1$. Suppose without loss of generality $\beta^2 > \beta^1$.

By the induction hypothesis, it follows there exists a unique $\beta^3 \in \mathbb{R}$, such that,

$$u(x'_1, \ldots, x'_{n+1}, x_{n+2}, \ldots) = u\left[\left[\begin{matrix} f_i(\beta^3) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^3) & \text{on} & n \\ f_{n+1}(\beta^2) & \text{on} & n+1 \\ x_j & \text{on} & j \not\in \{1, \ldots, n+1\} \end{matrix}\right]\right] \Leftrightarrow$$

$$\hat{u}(x'_1, \ldots, x'_{n+1}, x_{n+2}, \ldots) = \hat{u}\left[\left[\begin{matrix} f_i(\beta^3) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^3) & \text{on} & n \\ f_{n+1}(\beta^2) & \text{on} & n+1 \\ x_j & \text{on} & j \not\in \{1, \ldots, n+1\} \end{matrix}\right]\right].$$

By monotonicity $\beta^3 \in (\beta^1, \beta^2)$. To see this, notice that if $\beta^3 \geq \beta^2$ then

$$u\left[\left[\begin{matrix} f_i(\beta^3) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^3) & \text{on} & n \\ f_{n+1}(\beta^2) & \text{on} & n+1 \\ x_j & \text{on} & j \not\in \{1, \ldots, n+1\} \end{matrix}\right]\right] > u\left[\left[\begin{matrix} f_i(\beta^1) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^2) & \text{on} & n \\ f_{n+1}(\beta^2) & \text{on} & n+1 \\ x_j & \text{on} & j \not\in \{1, \ldots, n+1\} \end{matrix}\right]\right]$$

which is a contradiction. Similarly, if $\beta^3 \leq \beta^1$ then

$$u\left[\left[\begin{matrix} f_i(\beta^1) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^2) & \text{on} & n \\ f_{n+1}(\beta^2) & \text{on} & n+1 \\ x_j & \text{on} & j \not\in \{1, \ldots, n+1\} \end{matrix}\right]\right] > u\left[\left[\begin{matrix} f_i(\beta^3) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^3) & \text{on} & n \\ f_{n+1}(\beta^2) & \text{on} & n+1 \\ x_j & \text{on} & j \not\in \{1, \ldots, n+1\} \end{matrix}\right]\right].$$

Since $(n, n+1) \in G$ and u and $\hat{u}$ exhibit the properties monotonicity and solvability, it follows there exists an interval $\beta^4 \in \mathbb{R}$, such that,

$$u(x'_1, \ldots, x'_{n+1}, x_{n+2}, \ldots) = u\left[\left[\begin{matrix} f_i(\beta^3) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^4) & \text{on} & n \\ f_{n+1}(\beta^4) & \text{on} & n+1 \\ x_j & \text{on} & j \not\in \{1, \ldots, n+1\} \end{matrix}\right]\right] \Leftrightarrow$$

$$\hat{u}(x'_1, \ldots, x'_{n+1}, x_{n+2}, \ldots)$$

$$= \hat{u}\left(\left[\begin{array}{lll} f_i(\beta^3) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^4) & \text{on} & n \\ f_{n+1}(\beta^4) & \text{on} & n+1 \\ x_j & \text{on} & j \notin \{1, \ldots, n+1\} \end{array}\right]\right).$$

By monotonicity $\beta^4 \in (\beta^3, \beta^2)$. To see this, notice that if $\beta^4 \geq \beta^2$ then

$$u\left(\left[\begin{array}{lll} f_i(\beta^3) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^4) & \text{on} & n \\ f_{n+1}(\beta^4) & \text{on} & n+1 \\ x_j & \text{on} & j \notin \{1, \ldots, n+1\} \end{array}\right]\right) > u\left(\left[\begin{array}{lll} f_i(\beta^3) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^3) & \text{on} & n \\ f_{n+1}(\beta^2) & \text{on} & n+1 \\ x_j & \text{on} & j \notin \{1, \ldots, n+1\} \end{array}\right]\right)$$

which is a contradiction. And, similarly, if $\beta^4 \leq \beta^3$, then

$$u\left(\left[\begin{array}{lll} f_i(\beta^3) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^3) & \text{on} & n \\ f_{n+1}(\beta^2) & \text{on} & n+1 \\ x_j & \text{on} & j \notin \{1, \ldots, n+1\} \end{array}\right]\right) > u\left(\left[\begin{array}{lll} f_i(\beta^3) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta^4) & \text{on} & n \\ f_{n+1}(\beta^4) & \text{on} & n+1 \\ x_j & \text{on} & j \notin \{1, \ldots, n+1\} \end{array}\right]\right).$$

Reiterating the same operation we obtain two sequences: $\beta_1 < \beta_3 < \ldots < \beta_{2i+1} < \ldots$ and $\beta_2 > \beta_4 > \ldots > \beta_{2i} > \ldots$. Both sequences are bounded above by β_2 and below by β_1, and so the first sequence has a least upper bound, $\underline{b}$, and the second sequence has a greatest lower bound, $\bar{b}$. Furthermore by the MS property there exists a unique β for which

$$u(x'_1, \ldots, x'_{n+1}, x_{n+2}, \ldots) = u\left(\left[\begin{array}{lll} f_i(\beta) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta) & \text{on} & n \\ f_{n+1}(\beta) & \text{on} & n+1 \\ x_j & \text{on} & j \notin \{1, \ldots, n+1\} \end{array}\right]\right)$$

and

$$\hat{u}(x'_1, \ldots, x'_{n+1}, x_{n+2}, \ldots) = \hat{u}\left(\left[\begin{array}{lll} f_i(\beta) & \text{on} & i \in \{1, \ldots, n-1\} \\ f_n(\beta) & \text{on} & n \\ f_{n+1}(\beta) & \text{on} & n+1 \\ x_j & \text{on} & j \notin \{1, \ldots, n+1\} \end{array}\right]\right).$$

Once more by the MS property it follows that $\underline{b} = \beta = \overline{b}$. Thus by the induction argument applied to I, we have for each $\mathbf{x}$, $\mathbf{x}' \in \mathbf{X}$, there exists a *unique* $\beta_{\mathbf{x}}$, $\beta_{\mathbf{x}'} \in \mathbb{R}$ such that

$$u(\mathbf{x}) = u([f_i(\beta_{\mathbf{x}}) \quad \text{on} \quad i \in \mathbb{Z}]) \Leftrightarrow \hat{u}(\mathbf{x}) = \hat{u}([f_i(\beta_{\mathbf{x}}) \quad \text{on} \quad i \in \mathbb{Z}])$$
$$u(\mathbf{x}') = u([f_i(\beta_{\mathbf{x}'}) \quad \text{on} \quad i \in \mathbb{Z}]) \Leftrightarrow \hat{u}(\mathbf{x}') = \hat{u}([f_i(\beta_{\mathbf{x}'}) \quad \text{on} \quad i \in \mathbb{Z}]).$$

Hence we have

$$u(\mathbf{x}) \geq u(\mathbf{x}') \Leftrightarrow \beta_{\mathbf{x}} \geq \beta_{\mathbf{x}'} \Leftrightarrow \hat{u}(\mathbf{x}) \geq \hat{u}(\mathbf{x}').$$

This completes the proof.

4. Applications

An application of the result in the context of intertemporal consumption was already mentioned in Section 1. To conclude this note we mention below two other instances of potential applications.

Demand theory. Let $\mathbf{X} = \mathbb{R}^n_+$ be the consumption set where each X_i is the consumption set of commodity i. Let good 1 be the 'numeraire' and for every $i = 2, \ldots n$, suppose that $(0, 1) \in G$. Then, to verify whether the continuous weak orders $\succcurlyeq$ and $\hat{\succcurlyeq}$ are the same it is enough to check their conditional agreement on two dimensional consumption bundles.

Contingent consumption with a finite set of states. Let S be a set of states of nature and denote by $\mathbf{X} = \prod_{s \in S} X_s$, where each $X_s \subset \mathbb{R}^L_+$, the contingent consumption set. Suppose that, for every $s \in S$, $(s, s + 1) \in G$.

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References

Grant, S., Karni E., 2000. A theory of quantifiable beliefs. Unpublished manuscript.