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Incorporating fairness in generalized games of matching pennies

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We examine individual behavior in generalized games of matching pennies. We have three main findings. First, individuals cooperate in these games; that is, they systematically select strategies that lead both players to obtain higher expected payoffs than in a Nash equilibrium. Second, existing models that assume altruistic preferences do not explain the cooperative behavior in these games. Third, among the main models in the extant literature, the only one that predicts the observed behavior is the quantal response equilibrium.

Key words fairness, games, mixed-strategy, matching-pennies, quantal-response

JEL classification C72, C9

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1 Introduction

In the present paper we examine experimental behavior in "generalized games of matching pennies," a class of two-by-two games with no pure strategy Nash equilibria described in Figure 1. Prior work has shown that the quantal response model fits the data better than Nash equilibrium or noisy Nash equilibrium (McKelvey and Palfrey 1995; Ochs 1995; Goeree and Holt 2001; Goeree, Holt, and Palfrey 2003).

The existing papers have emphasized that the quantal response model is preferred because it models individuals as being boundedly rational. In the present paper we discuss a different property of the quantal response model: it is the only existing model that leads to cooperative behavior and outcomes in the generalized game of matching pennies. What is most surprising is that models that alter individuals' preferences to account for kindness, fairness, difference aversions and social welfare (e.g. Fehr and Schmidt 1999; Charness and Rabin 2002) cannot explain cooperation in these games.

An attractive characteristic of the class of games in Figure 1 is that players' risk preferences do not affect the set of equilibria.[1] To see this, notice that there are only three payoffs: 0, 1, and a. von Neumann–Morgenstern utilities can be normalized at two values: $u(0) = 0$ and $u(1) = 1$. Therefore, considering risk preferences only re-scales the parameter a.

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[1] By risk preference we simply mean "preferences over lotteries."

		Column	
		L (q)	R $(1-q)$
Row	u (p)	a, 0	0, 1
	d $(1-p)$	0, 1	1, 0

Figure 1 Generalized game of matching pennies, where $a > 0$.

To obtain some intuition for the results, we will show informally how starting from $a = 1$, when a is increased, the payoff for both players increases in the quantal response model but not necessarily with the other models. In a quantal response equilibrium, individuals select each action as an increasing function of its expected payoff. When $a = 1$, in equilibrium, Row chooses u with probability $p = 1/2$ and Column chooses R with probability $q = 1/2$; Row obtains the same expected payoff from u as from d, and Column obtains the same expected payoff from R as from L. An increase in a has four distinct effects. First, the direct effect (given Column's strategy $q = 1/2$) increases the expected payoff to Row of selecting u. Second, the quantile response effect on Row is for Row to select u more often; that is, p increases above 1/2 (but not to 1). Third, given that $p > 1/2$, R now gives Column the highest expected payoff and, hence, the quantile response for Column is to select R more often, but again not with certainty. Therefore, the expected payoff for Column increases. Fourth, the change in strategy by Column has a negative effect for Row for selecting u, which in turn will mitigate Row's increase in p; however, this is a second order effect, which, as we will demonstrate formally, does not fully offset the other effects and, therefore, both players obtain higher payoffs as a increases.

In contrast to the quantal response equilibrium, in a Nash equilibrium, a player's mixed strategy does not depend on his or her own payoffs. Because Row's mixed strategy does not depend on a, an increase in a does not change p and, hence, Column's payoff. When players are altruistic, in a mixed strategy Nash equilibrium, when a increases, Column's altruistic payoff from L increases. Therefore, to insure that the column player mixes, the Row players needs to select d with higher frequency, leading both players to lower payoffs!

In Section 2 we summarize the experimental findings for the generalized game of matching pennies. In Section 3 we describe the quantal response equilibrium and show that its comparative statics are consistent with the experimental findings. We then discuss other models in Section 4 and show how they all lead to predictions inconsistent with the experimental results. The Appendix contains all proofs.

2 Stylized facts

In the unique Nash equilibrium of the asymmetric game of matching pennies, Row selects u with probability $p = 1/2$ and Column selects L with probability $q = 1/(1 + a)$. In particular, the parameter a (the own payoff) does not affect the equilibrium mixed strategy of Row. The experimental evidence summarized in Table 1 contradicts this prediction. Column p^A denotes the frequency with which Row selected action u. For each of the three sets of experiments (i.e. Ochs 1995; McKelvey, Palfrey, and Weber 2000; Goeree and Holt 2001) the probability that Row selects u, p^A, is increasing in Row's payoff from selecting u, a.

Table 1 provides further evidence against the predictions of the Nash equilibrium. Column π_R is the expected payoff for Row given that the players select Nash equilibrium strategies. Column π_R^A is the expected payoff for Row given the actual strategies selected by the players. When $a \neq 1$ players select strategies that lead to higher expected payoffs for the Row player than in a Nash equilibrium. Similarly, Column also obtains higher expected payoffs than in a Nash equilibrium. For the experiment with $a = 7$, Row obtains 27 percent more than the Nash expected payoffs, whereas Column obtains 63 percent more than in a Nash equilibrium.

Therefore, the experimental evidence strongly points to individuals cooperating. These results might not appear that surprising given the extensive evidence of altruistic behavior observed in games (Fehr and Schmidt 1999; Charness and Rabin 2002). For this reason, Rabin (1993), Fehr and Schmidt (1999) and Charness and Rabin (2002) have developed game-theoretic equilibrium concepts that account for altruism. What we find surprising is that these models do not predict cooperative behavior in the generalized game of matching pennies, and are therefore inconsistent with our stylized facts.

We consider first a model of boundedly rational players (the quantal response equilibrium), with comparative statics consistent with the experimental results. It is already known that the quantal response equilibrium "fits" experimental data better than the Nash equilibrium (McKelvey and Palfrey 1995). However, it has not been pointed out before that, for the generalized game of matching pennies, the quantal response equilibrium predicts that individuals act cooperatively.

3 The quantal response equilibrium

Consider a two-player game in normal form (S_R, S_C, π_R, π_C), where for $i = R, C$, S_i is the finite set of actions for player i, and $\pi_i(s_i, s_j)$ is the payoff for player i when he or she selects action $s_i \in S_i$ and player j selects action $s_j \in S_j$. Let $\Delta(S_i)$ denote the set of probability distributions (i.e. mixed strategies) of player i. For each mixed strategy $\sigma_i \in \Delta(S_i)$ and each action $s_i \in S_i, \sigma_i(s_i) \geq 0$ denotes the probability that player i will select the action $s_i \in S_i$ in that mixed strategy. Naturally, $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$. With slight abuse of notation, s_i also denotes the degenerate mixed strategy for which the action s_i is selected with probability one. We set $\pi_i(\sigma_i, \sigma_j) = \sum_{s_i \in S_i} \sum_{s_j \in S_j} \pi_i(s_i, s_j)\sigma_i(s_i)\sigma_j(s_j)$ to be the expected payoff for player i when choosing his or her action according to the mixed strategy σ_i given that player j is selecting actions according to the mixed strategy σ_j.

McKelvey and Palfrey (1995) formulate a theory in which players choose their actions based on relative expected utilities and assume other players do so as well. According to McKelvey and Palfrey, such behavior arises as a result of players choosing their strategies with error (embodying a form of "bounded rationality"). The quantal response

Table 1 Summary of the experimental evidence

Experiment	Treatment	a	p	q	p^A	q^A	π_R	π_C	π_R^A	π_C^A	n	p-value
Goeree and Holt (2001)	RA	0.1	0.5	0.91	0.08	0.8	0.09	0.5	0.19	0.75	25	
Goeree and Holt (2001)	SM	1	0.5	0.5	0.48	0.48	0.5	0.5	0.50	0.50	50	0.001
Goeree and Holt (2001)	AM	7	0.5	0.125	0.96	0.16	0.875	0.5	1.08	0.81	25	0.000
Ochs (1995)	1	1	0.5	0.5	0.48	0.48	0.5	0.5	0.50	0.50	1536	
Ochs (1995)	3	4	0.5	0.2	0.54	0.34	0.8	0.5	1.04	0.52	512	0.002
Ochs (1995)	2	9	0.5	0.1	0.59	0.26	0.9	0.5	1.68	0.55	448	0.026
McKelvey, Palfrey, and Weber (2000)	D	4	0.5	0.2	0.55	0.328	0.8	0.5	1.024	0.52	600	0.014
McKelvey, Palfrey, and Weber (2000)	A	9	0.5	0.1	0.64	0.26	0.9	0.5	1.67	0.57	1800	0.000

The Nash equilibrium strategies are (p, q), while the actual frequencies with which u and L are selected are (p^A, q^A). Similarly, the expected payoffs for the row and column players are (π_R, π_C) when individuals select the Nash equilibrium strategies and (π_R^A, π_C^A) when individuals play according to the actual frequencies. The p-value denotes the probability with which the the row players select u with the same frequency as in the treatment with a smaller value of a. For instance, for Georee and Holt SM, the p-value is the probability with which individuals select u with the same frequency as in treatment RA. McKelvey, Palfrey, and Weber (2000) do not run an experiment with $a = 1$, so the p-value is computed under the assumption that when $a = 1$, the row players select u with probability 0.5. The p-value corresponds to a two-sided test; p-values for the one-sided test (which is all one needs to show that p is increasing in a) are half this size.

equilibrium is then obtained as the fixed point of this process. For a logit specification of the error structure they derive the following expressions for the quantal response function and associated equilibrium.

Definition 1 Fix $\lambda \geq 0$. The logistic quantal response function of player i to the mixed strategy σ_j, denoted by $LQR_i(\sigma_j)$, is given by

$$LQR_i(\sigma_j)(s_i) = \frac{e^{\lambda \pi_i(s_i,\sigma_j)}}{\sum_{\hat{s}_i \in S_i} e^{\lambda \pi_i(\hat{s}_i,\sigma_j)}}.$$

The logit equilibrium is a pair of mixed strategies (σ_R^L, σ_C^L), such that, $\sigma_R^L = LQR_R(\sigma_C^L)$ and $\sigma_C^L = LQR_C(\sigma_R^L)$.

When $\lambda = 0$, a player selects each action with equal probability. In the limit $\lambda \to \infty$, the probability that the player selects the action with the highest payoff tends to 1, and the equilibrium converges to a Nash equilibrium. Hence, the parameter λ may be interpreted as the degree of rationality of the player, where larger values of λ correspond to individuals who are more rational (or, more strictly speaking, more likely to select a best response). Of course, higher rationality by all players does not lead to higher payoffs, as is shown below.

Proposition 1 states that the logit equilibrium is consistent with the stylized facts discussed in Section 2.

Proposition 1 *Consider the generalized game of matching pennies (Figure 1). For all $\lambda > 0$, in the logit equilibrium, p is increasing in a, whereas q is decreasing in a. Furthermore, for all $a > 0$, $a \neq 1$, both players obtain higher expected payoffs than in the Nash equilibrium.*

A formal proof appears in the Appendix, but to provide some intuition let us start with $a = 1$; that is, the standard symmetric matching pennies. The logit equilibrium is the same as the Nash equilibrium with $p = q = 1/2$, because for each player, given his or her opponent's behavior, each of the two available actions yields the same expected payoff; namely, 1/2. If a was increased and Column's behavior remained unchanged, then the payoff to Row of playing u would be greater than that from playing d and so the logistic quantal response would entail a mixed strategy in which Row plays u with a probability $p > 1/2$ (but unlike the Nash best response, still less than 1). If Row were playing u with a probability $p > 1/2$, however, Column's expected payoff from selecting R would now be higher than from selecting L, and in turn Column's logistic quantal response would entail a mixed strategy with $q < 1/2$ (but again, unlike a Nash response, would still be greater than 0). Although this change in Column's mixed strategy reduces the expected payoff of Row playing u versus d, the algebra in the proof establishes that $q > 1/(1 + a)$, which in turn means that this strategic response of Column does not fully offset the direct effect on Row's payoff from the original increase in a. Furthermore, because in the new logit equilibrium $p > 1/2$ and $1/(1 + a) < q < 1/2$, it readily follows that the expected payoffs to Row and Column are greater than what they would be if they were playing a Nash equilibrium. Using an analogous argument it can readily be shown that were a to be reduced below 1, in the new logit equilibrium we would have $p < 1/2$ and $1/(1 + a) > q > 1/2$, with expected payoffs to both higher than their respective Nash equilibrium payoffs.

The direct effect of moving a away from 1 is to induce Row to select one of his or her actions with a higher probability than the other. By being "less random" in his or her action choice (notice that the entropy of Row's quantal best response has fallen), Row has conferred a benefit on Column. Column "cooperates" by not fully exploiting this opportunity. That is, Column's quantal response induces Column to place more weight on his or her now preferred action but, unlike a best response, the error structure in Column's action choice leads Row to expect Column to choose the action that benefits Row with sufficiently high probability to have Row's quantal response still involve an unequal weighting in choosing between u and d.

In a Nash equilibrium, the parameter a does not affect the likelihood with which the row player selects u, and, therefore, Column's payoffs do not change with a. Hence, the bounded rationality of the players stemming from the possible errors they might make in choosing an action and their awareness that their opponents might make similar mistakes leads to a logit equilibrium of the asymmetric matching pennies game in which both players choose mixed strategies that are more cooperative than if they were both playing a Nash equilibrium with an attendant outcome in which both players have a higher expected payoff than if they were playing a Nash equilibrium.

4 Alternative theories

4.1 Fairness

Individuals might cooperate if their utility functions depend not only on the payoff of the game but also on their kindness and the perceived kindness of others. These ideas are formalized in Rabin (1993) and are described below. For each $\sigma_j \in \Delta(S_j)$, let

$$\Pi(\sigma_j) \equiv t\{(\pi_i(\sigma_i, \sigma_j), \pi_j(\sigma_j, \sigma_i)) : \sigma_i \in \Delta(S_i)\};$$

that is, $\Pi(\sigma_j)$ is the set of pair of payoffs for both players attainable by player i when player j is choosing $\sigma_j \in \Delta(S_j)$. Let $\pi_j^h(\sigma_j)$ (respectively, $\pi_j^\ell(\sigma_j)$) denote player j's highest (respectively, lowest) payoff among points that are Pareto efficient in $\Pi(\sigma_j)$, and let the *equitable* payoff be

$$\pi_j^e(\sigma_j) = \left[\pi_j^h(\sigma_j) + \pi_j^\ell(\sigma_j)\right]/2.$$

Finally, let $\pi_j^{\min}(\sigma_j)$ be the worst possible payoff for player j in the set $\Pi(\sigma_j)$.

Definition 2 Suppose player i selects action $s_i \in S_i$ and believes that player j selects the mixed strategy $\sigma_j \in \Delta(S_j)$. Then, the kindness of i's action to player j' strategy is

$$f_i(s_i, \sigma_j) \equiv \begin{cases} \dfrac{\pi_j(\sigma_j, s_i) - \pi_j^e(\sigma_j)}{\pi_j^h(\sigma_j) - \pi_j^{\min}(\sigma_j)} & \text{if } \pi_j^h(\sigma_j) > \pi_j^{\min}(\sigma_j), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3 Player i's *belief* about how kind player j is being to player i given player i's belief that player j is playing the mixed strategy $\sigma_j \in \Delta(S_j)$ and his or her belief that player j believes player i is choosing the mixed strategy $\sigma_i \in S_i$, is given by

$$\tilde{f}_j(\sigma_j, \sigma_i) = \frac{\pi_i(\sigma_i, \sigma_j) - \pi_i^e(\sigma_i)}{\pi_i^h(\sigma_i) - \pi_i^{\min}(\sigma_i)},$$

if $\pi_i^h(\sigma_i) > \pi_i^{\min}(\sigma_i)$, and 0 otherwise.

Following Rabin (1993), in order to incorporate "fairness" into the payoffs, we assume that an individual maximizes a convex combination of his or her *material* payoff and his or her *expectation* of the product of his or her *belief about how kind player j is being to* the individual and the individual's *kindness to player j plus* 1. That is, suppose player i believes that player j is playing the mixed strategy $\sigma_j \in \Delta(S_j)$ and, furthermore, i believes that j believes that i is playing the mixed strategy $\sigma_i \in \Delta(S_i)$, then i's expected payoff from playing the pure strategy s_i is given by

$$U_i(s_i, \sigma_j, \sigma_i) = (1-\alpha)\pi_i(s_i, \sigma_j) + \alpha \tilde{f}_j(\sigma_j, \sigma_i)[1 + f_i(s_i, \sigma_j)],$$

where $\alpha \in (0, 1)$.

Definition 4 (σ_1^F, σ_2^F) is a *fairness equilibrium* if for $i = 1, 2$, $j \neq i$,

$$\sigma_i^F(s_i) > 0 \Rightarrow U_i\left(s_i, \sigma_j^F, \sigma_i^F\right) \geq U_i\left(s_i', \sigma_j^F, \sigma_i^F\right) \text{ for all } s_i' \in S_i.$$

Proposition 2 *The unique fairness equilibrium of the asymmetric matching pennies game is*

$$\left(\frac{1}{2}, \frac{2-\alpha}{2\left[(1-\alpha)\,a + 1\right]}\right).$$

This result contradicts the experimental evidence presented in Table 1 that shows that p is increasing in a. Instead, Row mixes in the fairness equilibrium the same as if the players selected strategies according to the Nash equilibrium. Therefore, Column obtains exactly the same payoff as in a Nash equilibrium.

4.2 Social preferences

An alternative approach to "fairness" is to assume that a player has a direct regard for other players' material payoff, as well as his or her own. We shall say an individual has social preferences (see Charness and Rabin 2002) if the utility for individual i when player i selects action s_i and player j selects action s_j is $U_i(s_i, s_j) = u_i(\pi_i(s_i, s_j), \pi_j(s_j, s_i))$, where

$$u_i(x_i, x_j) = (1 - \rho r(x_i, x_j) - \tau s(x_i, x_j))x_i + (\rho r(x_i, x_j) + \tau s(x_i, x_j))x_j,$$
$$r(x_i, x_j) = \begin{cases} 1 & \text{if } x_i > x_j \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad s(x_i, x_j) = \begin{cases} 1 & \text{if } x_i < x_j \\ 0 & \text{otherwise.} \end{cases}$$

		Column	
		L (q)	R ($1-q$)
Row	u (p)	τa $a(1-\rho)$	$1-\rho$ τ
	d ($1-p$)	$1-\rho$ τ	τ $1-\rho$

Figure 2 Generalized game of matching pennies with social references, where $a > 0$.

That is, we have

$$\begin{aligned} u_i(x_i, x_j) &= x_i + r(x_i, x_j)\rho(x_j - x_i) + s(x_i, x_j)\tau(x_j - x_i) \\ &= x_i - \rho \max(0, x_i - x_j) + \tau \max(0, x_j - x_i). \end{aligned}$$

Definition 5 A *social-welfare equilibrium* is a Nash equilibrium of the game with preferences (U_R, U_C).

This model was first considered by Fehr and Schmidt (1999) with the additional restriction that $\rho \le -\tau$ and $0 \le \rho \le 1$. Charness and Rabin (2002) label preferences depending on the parameter values as follows:

- Competitive preferences correspond to $\tau \le \rho \le 0$
- Difference aversion preferences correspond to $\tau < 0 < \rho < 1$
- Social-welfare preferences correspond to $1 \ge \rho \ge \tau > 0$.

They interpret their experimental evidence as supporting the social-welfare preferences. However, as shown in Proposition 3, none of the preferences considered in Charness and Rabin are consistent with the experimental evidence in the generalized game of matching pennies.

Proposition 3 *Consider the generalized game of matching pennies (Figure 2). In a mixed equilibrium social-welfare equilibrium, one of the players obtains a payoff that is lower or equal to the Nash payoff.*

5 Conclusion

In the present paper we have demonstrated a surprising property of the generalized game of matching pennies. Equilibrium concepts that attempt to model cooperative behavior do not predict that individuals would be able to cooperate in this game. However, a model considering how individuals select actions with errors predicts that individuals cooperate. Furthermore, experimental results are consistent with this prediction.

We have defined cooperative behavior as the players choosing mixed strategies that differ from what they would have chosen if they had been playing a Nash equilibrium, which leads to both players obtaining payoffs that are higher than in a Nash equilibrium.

Clearly, the quantal response equilibrium does not possess this property for all games. For instance, consider a game that has a unique Nash equilibrium that is Pareto efficient. Then, any equilibrium model in which players do not always select the best response leads to payoffs that are lower than in a Nash equilibrium. What this suggests, however, is that part of the success of the quantal response equilibrium in analyzing the game of matching pennies might be due to the fact that for this class of games the quantal response equilibrium corresponds to cooperative behavior.

Appendix

Remark 1 To describe the experiments in Goeree and Holt (2001) as generalized matching pennies games, we subtracted 40 from each player's payoff and then divided all payoffs by 40. Subtracting a constant from all cells does not change the quantal response, the fairness, or the social welfare equilibrium. Multiplying the payoffs by a constant changes the values of λ in the quantal response equilibrium, and the values of ρ and τ in the social-welfare equilibrium. This is one reason why comparisons should be made within different treatments of the same experiment (i.e. Goeree and Holt (AM) and Goeree and Holt (RS)) and not across experiments.

Lemma 1 *When $a > 1$, both players obtain a payoff higher than in a Nash equilibrium if and only if (i) $p > 1/2$ and $q \in (\frac{1}{1+a}, 1/2)$ or (ii) $p \in (\frac{1}{1+a}, 1/2)$ and $q > 1/2$; when $a < 1$, both players obtain a payoff higher than in a Nash equilibrium if and only if (i) $p \in (1/2, \frac{1}{1+a})$ and $q < 1/2$* or (ii) *$p < 1/2$ and $q \in (1/2, \frac{1}{1+a})$.*

PROOF: Under the Nash equilibrium, the expected payoff of the row player is $\frac{a}{1+a}$, whereas the expected payoff of the column player is 1/2. In order for the Row player to obtain a higher payoff than in a Nash equilibrium,

$$\begin{aligned} &pqa + (1-p)(1-q) > \tfrac{a}{1+a} \\ \Leftrightarrow\ &p(qa - 1 + q) > \tfrac{a}{1+a} - 1 + q = \tfrac{1}{1+a}[qa - 1 + q]. \end{aligned}$$

Therefore, the Row player obtains a higher payoff than in a Nash equilibrium if $qa - 1 + q > 0$ and $p > \frac{1}{1+a}$ or $qa - 1 + q < 0$ and $p < \frac{1}{1+a}$. Similarly, we show that the Row player obtains a payoff higher than in a Nash equilibrium if $pa - 1 + p > 0$ and $q > \frac{1}{1+a}$ or $pa - 1 + p < 0$ and $q < \frac{1}{1+a}$. This leads us to consider four cases:

(i) $a > 1$ and $p > 1/2$. Then, $\pi_C(p, R) = p > 1/2 > 1 - p = \pi_C(p, L)$. Therefore, the column players obtain a higher payoff than in a Nash equilibrium if and only if $q < 1/2$. Because $pa - 1 + p > 0$, Row obtains a higher payoff than in a Nash equilibrium if and only if $q > \frac{1}{1+a}$.

(ii) $a > 1$ and $p < 1/2$. Then, $\pi_C(p, R) = p < 1/2 < 1 - p = \pi_C(p, L)$. Thus, the column players obtain a higher payoff than in a Nash equilibrium if and only if $q > 1/2$. Because $qa - 1 + q > 0$, Row obtains a higher payoff than in a Nash equilibrium if and only if $p > \frac{1}{1+a}$.

(iii) $a < 1$ and $p > 1/2$. Then, $\pi_C(p, R) = p > 1/2 > 1 - p = \pi_C(p, L)$. Thus, the column players obtain a higher payoff than in a Nash equilibrium if and only if $q < 1/2$. Because $qa - 1 + q < 0$, Row obtains a higher payoff than in a Nash equilibrium if and only if $p < \frac{1}{1+a}$.

(iv) $a < 1$ and $p < 1/2$. Then, $\pi_C(p, R) = p < 1/2 < 1 - p = \pi_C(p, L)$. Thus, the column players obtain a higher payoff than in a Nash equilibrium if and only if $q > 1/2$. Because $pa - 1 + p < 0$, Row obtains a higher payoff than in a Nash equilibrium if and only if $q < \frac{1}{1+a}$. □

Proposition 1 *For all $\lambda > 0$, in the quantal response equilibrium, p is increasing in a, q is decreasing in a, and both players obtain higher expected payoffs than in the Nash equilibrium.*

PROOF: The payoffs are rewritten as $u(0)=0$, $u(1)=1$, and $u(a)=\alpha$. Let

$$f(p,q;\alpha,\lambda) = \frac{e^{\lambda\alpha q}}{e^{\lambda\alpha q}+e^{\lambda(1-q)}} - p$$
$$g(p,q;\alpha,\lambda) = \frac{e^{\lambda(1-p)}}{e^{\lambda(1-p)}+e^{\lambda p}} - q.$$

Then, a quantal response equilibrium (p^L, q^L) satisfies

$$f(p^L,q^L;\alpha,\lambda) = 0, \quad g(p^L,q^L;\alpha,\lambda) = 0.$$

By the implicit function theorem,

$$\begin{pmatrix} f_p & f_q \\ g_p & g_q \end{pmatrix}\begin{pmatrix} p_\alpha \\ q_\alpha \end{pmatrix} = \begin{pmatrix} -f_\alpha \\ -g_\alpha \end{pmatrix},$$

where $f_p = g_q = -1, g_\alpha = 0$,

$$\begin{aligned}
f_q &= \frac{\alpha\lambda e^{\lambda\alpha q}\left(e^{\lambda\alpha q}+e^{\lambda(1-q)}\right)+\left(-\lambda\alpha e^{\lambda\alpha q}+\lambda e^{\lambda(1-q)}\right)e^{\lambda\alpha q}}{\left(e^{\lambda\alpha q}+,e^{\lambda(1-q)}\right)^2}\\
&= \frac{(1+\alpha)\lambda e^{\lambda\alpha q}e^{\lambda(1-q)}}{\left(e^{\lambda\alpha q}+e^{\lambda(1-q)}\right)^2} > 0\\
g_p &= \frac{-\lambda e^{\lambda(1-p)}\left(e^{\lambda(1-p)}+e^{\lambda p}\right)+\left(\lambda e^{\lambda(1-p)}-\lambda e^{\lambda p}\right)e^{\lambda(1-p)}}{\left(e^{\lambda(1-p)}+e^{\lambda p}\right)^2}\\
&= \frac{-2\lambda e^{\lambda p}e^{\lambda(1-p)}}{\left(e^{\lambda(1-p)}+e^{\lambda p}\right)^2} < 0\\
f_\alpha &= \frac{\lambda q e^{\lambda\alpha q}\left(e^{\lambda\alpha q}+e^{\lambda(1-q)}\right)-\lambda q e^{\lambda\alpha q}e^{\lambda\alpha q}}{\left(e^{\lambda\alpha q}+e^{\lambda(1-q)}\right)^2}\\
&= \frac{\lambda q e^{\lambda(1-q)}e^{\lambda\alpha q}}{\left(e^{\lambda\alpha q}+e^{\lambda(1-q)}\right)^2} > 0.
\end{aligned}$$

By Cramer's rule,

$$p_\alpha = \frac{f_\alpha}{1-f_q g_p} > 0$$
$$q_\alpha = \frac{f_\alpha g_p}{1-f_q g_p} < 0.$$

Notice that because $f_q > 0$ and $g_p < 0, 1-f_q g_p > -f_q g_p$. Therefore,

$$q_\alpha = \frac{f_\alpha g_p}{1-f_q g_p} > \frac{f_\alpha g_p}{-f_q g_p} = -\frac{f_\alpha}{f_q}.$$

Table 2 Equitable payoffs for generalized game of matching pennies

j	σ_j	π_j^h	$\pi_j^\ell = \pi_j^{\min}$	π_j^e
Row	p	$\max(ap, 1-p)$	$\min(ap, 1-p)$	$(ap-p+1)/2$
Column	q	$\max(q, 1-q)$	$\min(q, 1-q)$	$1/2$

Substituting for the values of f_α and f_q that we obtained by the implicit function theorem gives

$$\begin{aligned} q_\alpha &> -f_\alpha \times (1/f_q) \\ &= -\frac{\lambda q e^{\lambda(1-q)} e^{\lambda\alpha q}}{\left(e^{\lambda\alpha q} + e^{\lambda(1-q)}\right)^2} \times \frac{\left(e^{\lambda\alpha q} + e^{\lambda(1-q)}\right)^2}{(1+\alpha)\lambda e^{\lambda\alpha q} e^{\lambda(1-q)}} \\ &= -\frac{q}{1+\alpha}. \end{aligned}$$

Notice that when $a = 1$, $\alpha = 1$, $p^L(1) = q^L(1) = \frac{1}{\alpha+1} = \frac{1}{2}$. Therefore, when $\alpha = 1$, $q_\alpha = \frac{q}{1+\alpha}$ and $\frac{dq_L}{da} > \frac{d}{da}(\frac{1}{1+a})$. In general, whenever $q^L \geq \frac{1}{1+a}$, $\frac{dq^L}{da} > \frac{d}{da}(\frac{1}{1+a})$. It follows that for all $a > 1$, $q^L > \frac{1}{1+\alpha}$. Similarly, note that when $a=1$, $p=1/2$ and $q=1/(1+a)$. Because $p_\alpha > 0$ and $q_\alpha < 0$, it follows that for $a>1$, $p^L > 1/2$ and $q^L < 1/2$. Therefore, by Lemma 1 both players obtain a higher payoff than in a Nash equilibrium. The same argument is used for the case when $a < 1$. □

Proposition 2 *The unique fairness equilibrium of the asymmetric matching pennies game is* $(\frac{1}{2}, \frac{2-\alpha}{2[(1-\alpha)a+1]})$.

PROOF: From the payoff matrix in Figure 1 we obtain the results in Table 2.

Plugging these values into the definition of $f_i(\cdot,\cdot)$ we calculate:

$$f_R(u,q) = \frac{\pi_C(q,u) - \pi_C^e(q)}{\pi_C^h(q) - \pi_C^{\min}(q)} = \frac{1-q-1/2}{|1-2q|} = \begin{cases} 1/2 & \text{if } q < 1/2 \\ 0 & \text{if } q = 1/2 \\ -1/2 & \text{if } q > 1/2 \end{cases};$$

$$f_R(d,q) = \frac{\pi_C(q,d) - \pi_C^e(q)}{\pi_C^h(q) - \pi_C^{\min}(q)} = \frac{q-1/2}{|1-2q|} = \begin{cases} -1/2 & \text{if } q < 1/2 \\ 0 & \text{if } q = 1/2 \\ 1/2 & \text{if } q > 1/2 \end{cases};$$

$$f_C(L,p) = \frac{\pi_R(p,L) - \pi_R^e(p)}{\pi_R^h(p) - \pi_R^{\min}(p)} = \frac{ap - (ap-p+1)/2}{|ap+p-1|} = \begin{cases} -1/2 & \text{if } p < \frac{1}{a+1} \\ 0 & \text{if } p = \frac{1}{a+1} \\ 1/2 & \text{if } p > \frac{1}{a+1} \end{cases};$$

$$f_C(R,p) = \frac{\pi_R(p,R) - \pi_R^e(p)}{\pi_R^h(p) - \pi_R^{\min}(p)} = \frac{(1-p)-(ap-p+1)/2}{|ap+p-1|} = \begin{cases} 1/2 & \text{if } p < \frac{1}{a+1} \\ 0 & \text{if } p = \frac{1}{a+1} \\ -1/2 & \text{if } p > \frac{1}{a+1} \end{cases}.$$

Plugging these values into the definition of $\tilde{f}_i(\cdot,\cdot)$ yields:

$$\tilde{f}_C(q,p) = \frac{paq + (1-p)(1-q) - (ap-p+1)/2}{|ap+p-1|} = \frac{(pa-1+p)(q-1/2)}{|ap+p-1|}$$

$$= \begin{cases} 1/2-q & \text{if } p < 1/(1+a) \\ 0 & \text{if } p = 1/(1+a) \\ q-1/2 & \text{if } p > 1/(1+a) \end{cases},$$

$$\tilde{f}_R(p,q) = \frac{p(1-q)+(1-p)q-1/2}{|1-2q|} = \frac{(1-2q)(p-1/2)}{|1-2q|}$$

$$= \begin{cases} p-1/2 & \text{if } q < 1/2 \\ 0 & \text{if } q = 1/2 \\ 1/2-p & \text{if } q > 1/2 \end{cases}.$$

Therefore, $(p,q) \in (0,1)^2$ is an equilibrium if

$$\text{Row}: U_R(u,q,p) - U_R(d,q,p) = 0 \quad (1)$$

$$\text{Column}: U_C(L,p,q) - U_C(R,p,q) = 0. \quad (2)$$

Equation (1) may be reexpressed as

$$(1-\alpha)[\pi_R(u,q) - \pi_R(d,q)] + \alpha\tilde{f}_C(q,p)[f_R(u,q) - f_R(d,q)] = 0 \quad (3)$$

and (2) becomes

$$(1-\alpha)[\pi_C(L,p) - \pi_C(R,p)] + \alpha\tilde{f}_R(p,q)[f_C(L,p) - f_C(R,p)] = 0. \quad (4)$$

Assuming $p > 1/(1+a)$, then we have by plugging in the appropriate values into (4):

$$\text{Column}: \begin{array}{ll} (1-\alpha)(1-2p) + \alpha(p-1/2) = 0 & \text{if } q < 1/2 \\ (1-\alpha)(1-2p) - \alpha(p-1/2) = 0 & \text{if } q \geq 1/2. \end{array}$$

If $\alpha \neq 2/3$, the only solution is $p = 1/2$. If instead we assume $p < 1/(1+a)$ we have

$$\text{Column}: \begin{array}{ll} (1-\alpha)(1-2p) - \alpha(p-1/2) = 0 & \text{if } q < 1/2 \\ (1-\alpha)(1-2p) + \alpha(p-1/2) = 0 & \text{if } q \geq 1/2 \end{array}$$

and, again, the only solution is $p = 1/2$.

So assume $p = 1/2$ or $\alpha = 2/3$ and $p > \frac{1}{1+a}$. Now consider $a > 1$. Because $p > 1/(1+a)$, (1) becomes

$$\text{Row}: \begin{array}{ll} (1-\alpha)(qa-1+q) + \alpha(q-1/2) = 0 & \text{if } q < 1/2 \\ (1-\alpha)(qa-1+q) - \alpha(q-1/2) = 0 & \text{if } q \geq 1/2, \end{array}$$

which yields, for $q < 1/2$, the solution

$$q = \frac{1-\alpha+1}{2[(1-\alpha)a+1]} < \frac{1}{2}.$$

For $q \geq 1/2$ the solution is

$$q = \frac{2-3\alpha}{2[(1-\alpha)(1+a)-\alpha]}.$$

For this value to be greater than or equal to 1/2 requires

$$2-3\alpha > (1-\alpha)(1+a)-\alpha \Leftrightarrow 2(1-\alpha) \geq (1-\alpha)(1+a).$$

This last inequality can never hold because $\alpha < 1$ and $a > 1$.

It remains to consider the case of $a < 1$. Because $p < 1/(1+a)$, (1) becomes

$$\text{Row}: \begin{array}{ll} (1-\alpha)(qa-1+q)+\alpha(1/2-q)=0 & \text{if } q < 1/2 \\ (1-\alpha)(qa-1+q)-\alpha(1/2-q)=0 & \text{if } q \geq 1/2. \end{array}$$

For $q < 1/2$ the solution

$$q = \frac{2-3\alpha}{2[(1-\alpha)(1+a)-\alpha]}$$

is in the open interval (0, 1/2) if

$$\begin{aligned} 2-3\alpha &< (1-\alpha)(1+a)-\alpha \\ \Leftrightarrow (1-\alpha) &< a(1-\alpha). \end{aligned}$$

Again, this last inequality cannot hold because $\alpha < 1$.
For $q \geq 1/2$ the solution

$$q = \frac{2-\alpha}{2[(1-\alpha)a+1]}$$

is always the interval [1/2, 1). □

Proposition 3 *In a mixed equilibrium social-welfare equilibrium, one of the players obtains a payoff that is lower or equal to the Nash payoff.*

PROOF: In a mixed strategy equilibrium the expected payoff for Row of u is the same as the expected payoff of d. This means that

$$q(a-\rho a)+(1-q)\tau = q\tau+(1-q)(1-\rho),$$

or

$$q = \frac{1-\tau-\rho}{1+a-2\tau-\rho-a\rho} = \frac{(1-\rho)-\tau}{(1+a)(1-\rho)-2\tau}.$$

Similarly, the expected payoff of Column for L is the same as the expected payoff for R:

$$p(\tau a)+(1-p)(1-\rho) = p(1-\rho)+(1-p)\tau,$$

or

$$p = \frac{1-\tau-\rho}{2-\tau-a\tau-2\rho} = \frac{(1-\rho)-\tau}{2(1-\rho)-\tau(1+a)}.$$

Note that

$$p = \frac{(1-\rho)-\tau}{2(1-\rho)-\tau(1+a)} > \frac{1}{2} \Leftrightarrow 2(1-\rho)-2\tau > 2(1-\rho)-\tau(1+a).$$

This is equivalent to: if $a > 1$, then $\tau > 0$ and if $a < 1$, then $\tau < 0$. Similarly,

$$p = \frac{(1-\rho)-\tau}{2(1-\rho)-\tau(1+a)} > \frac{1}{1+a} \Leftrightarrow (1-\rho)(1+a)-\tau(1+a) > 2(1-\rho)-\tau(1+a).$$

This is equivalent to: if $a > 1$, then $\rho < 1$ and if $a < 1$, then $\rho > 1$. Furthermore,

$$q = \frac{(1-\rho)-\tau}{(1+a)(1-\rho)-2\tau} < \frac{1}{2} \Leftrightarrow 2(1-\rho)-2\tau < (1+a)(1-\rho)-2\tau.$$

This is equivalent to: if $a > 1$, then $\rho < 1$ and if $a < 1$, then $\rho > 1$. Finally,

$$q = \frac{(1-\rho)-\tau}{(1+a)(1-\rho)-2\tau} > \frac{1}{1+a} \Leftrightarrow (1+a)(1-\rho)-2\tau < (1+a)(1-\rho)-\tau(1+a).$$

This is equivalent to: if $a > 1$ then $\tau < 0$ and if $a < 1$, then $\tau > 0$.

By Lemma 1, for both players to obtain a higher payoff, one of four possible cases is required:

- $a > 1$, $p > 1/2$ and $q \in (\frac{1}{1+a}, 1/2)$. This case is not possible because $p > 1/2$ implies that $\tau > 0$, while $q > \frac{1}{1+a}$ implies that $\tau < 0$.
- $a > 1$, $p \in (\frac{1}{1+a}, 1/2)$, and $q > 1/2$. This case is not possible because $p > \frac{1}{1+a}$ implies that $\rho < 1$, while $q > 1/2$ implies that $\rho > 1$.
- $a < 1$, $p \in (1/2, \frac{1}{1+a})$, and $q < 1/2$. This case is not possible because $p < \frac{1}{1+a}$ implies that $\rho < 1$, while $q < 1/2$ implies that $\rho > 1$.
- $a < 1$, $p < 1/2$, and $q \in (1/2, \frac{1}{1+a})$. This case is not possible because $p < 1/2$ implies that $\tau > 0$, while $q < \frac{1}{1+a}$ implies that $\tau < 0$.

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