Strategic Ambiguity in Multi-Stage Games with Almost Perfect Information.*

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Abstract

We propose a solution concept, consistent-planning equilibrium under ambiguity (CP-EUA), for two-player multi-stage games with almost perfect information. Players are neo-expected payoff maximizers. The associated (ambiguous) beliefs are revised by Generalized Bayesian Updating. Individuals take account of possible changes in their preferences by using consistent planning. We show that if there is ambiguity in the centipede game and players are sufficiently optimistic then it is possible to sustain 'cooperation' for many periods. Similarly, in a non-cooperative bargaining game we show that there may be delay in agreement being reached.

Keywords: optimism, neo-additive capacity, extensive-form games, dynamic consistency, consistent planning, centipede game.

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1 Introduction

The theory of decision making under uncertainty provides a wide range of representations that can accommodate behavioral biases, both for the case when the actual probabilities of events are known, for example, Prospect Theory (Kahneman and Tversky 1979), and for situations where no or only partial information about the probabilities of events are available, as in Choquet expected utility (Schmeidler 1989) or multiple prior approaches (Gilboa and Schmeidler 1989, Ghirardato, Maccheroni, and Marinacci 2004). Yet, none of these criteria for decision making under ambiguity has been that successfully implemented in a dynamic-game context where the ambiguity concerns the behavior of the opponent's strategy.[1] In our opinion, the main obstacles have been the difficult questions regarding:

- how much consistency should be required between the beliefs of players about their opponents' behavior and their actual behavior in order to justify calling a situation an (at least temporary) equilibrium; and,
- how much consistency to impose on dynamic strategies since all consequentialist updating rules for non-expected-utility representations essentially entail violations of dynamic consistency.[2]

In this paper, we propose an equilibrium concept for dynamic games with players that have non-neutral attitudes toward the strategic ambiguity they face by extending a notion of equilibrium for static games (Equilibrium under Ambiguity) studied in Eichberger and Kelsey (2014) and employing a general notion of updating for beliefs (Generalized Bayesian Updating) analyzed in Eichberger, Grant, and Kelsey (2007). Both concepts are general in that they can be applied to preference representations where beliefs are represented by capacities (Choquet expected utility, prospect theory) or by sets of multiple priors (Maxmin and α-maxmin expected utility). However, by applying these concepts to the *non extreme outcome (neo)*-expected utility representation axiomatized in Chateauneuf, Eichberger, and Grant (2007), enables us, with just two additional (unit-interval valued) parameters, to accommodate a wide range of attitudes that players may exhibit toward strategic ambiguity. The first parameter reflects the player's perception of strategic ambiguity by measuring that player's degree of confidence regarding their probabilistic beliefs about their opponent's behavior. The second

[1] There is a small literature on ambiguity in games in strategic form. Our earlier approach, Eichberger and Kelsey (2000), has its roots in Dow and Werlang (1994) and is similar to Marinacci (2000).

More recently, Riedel and Sass (2014) explore more general strategy notions ("Ellsberg strategies") in games of complete information and Azrieli and Teper (2011), Kajii and Ui (2005) and Grant, Meneghel, and Tourky (2016) study games of incomplete information with ambiguity about priors.

[2] An exception is the recursive multiple priors model of Epstein and Schneider (2003) that retains dynamic consistency with a consequentialist updating rule but at the cost of imposing stringent restrictions on what form the information structure may take.

measures a player's relative pessimistic versus optimistic attitudes toward this perceived strategic ambiguity.[3] Probabilistic beliefs are endogenously determined in equilibrium and are updated in the usual Bayesian way when new information arrives. With new information, however, a player's degree of confidence will change as well and, hence, the impact of a player's relative pessimistic versus optimistic attitudes toward the strategic ambiguity. This novel framework allows us to study the role of strategic ambiguity in games both under optimism and pessimism.[4]

In order to ensure dynamic consistency, we will adapt the notion of consistent planning proposed by Strotz (1955) and Siniscalchi (2011) for the game-theoretic context.[5] As we argue below, consistent planning finds behavioral support in a large literature in psychology on "self-regulation" Baumeister and Vohs (2004). Moreover, it allows us to work with the well-known backward induction methodology.

To demonstrate the potential of our approach, we apply it to multi-stage two-player games with almost perfect information.[6] It is within this context that many, if not most, deviations of human behavior from subgame perfect behavior have been noted. In particular, we show that a small degree of optimism in combination with some ambiguity induces equilibrium behavior which corresponds to behavior observed in experimental studies. As examples, we have chosen two of the most challenging cases from this class of games: the centipede game and the alternating-offer bargaining game. For the former, we provide a characterization of equilibria under ambiguity in terms of the perception of ambiguity and attitudes toward perceived ambiguity parameters. For the latter, we show that inefficient delays in bargaining may be the result of ambiguity about the other player's behavior. Though our framework allows us to derive equilibria by adapting well-known backward induction methods, the results reveal new channels of influence on behavior. In particular, the importance of some optimism in the face of uncertainty is highlighted, a channel of influence widely disregarded, since almost all preference representations under ambiguity have been axiomatized and analyzed for the case of pessimism only.

[3] Neo-expected utility is a special case of Choquet expected utility Schmeidler (1989) and of α-multiple priors expected utility Hurwicz (1951), Gul and Pesendorfer (2015). It can also be related to rank-dependent expected utility, Quiggin (1982).

[4] There is also a small earlier literature on extensive form games. Lo (1999) provides the first model treating ambiguity in extensive form games. Rothe (2011) proposes a generalization of subgame perfection for players with non-additive beliefs that is similar to the equilibrium concept we develop but for the most part the players in his model only exhibit pessimism toward the ambiguity they perceive there to be about the strategy choice of their opponents. All other papers deal with ambiguity in special cases: Eichberger and Kelsey (1999) and Eichberger and Kelsey (2004) study signaling games and,.more recently, Kellner and LeQuement (2015) cheap-talk games and Bose and Renou (2014) mechanism design questions with communication.

[5] Bose and Daripa (2009) employ a similar notion of consistent planning for their equlilibrium with players whose preferences have a multiple prior representation.

[6] Osborne and Rubinstein (1994) (p102) refer to this class as extensive games with perfect information and simultaneous moves.

2 Framework and Definitions

In this section we describe how we model ambiguity, updating and dynamic choice.

2.1 Ambiguous Beliefs and Expectations

For a typical two-player game let $i \in \{1,2\}$, denote a generic player. We shall adopt the convention of referring to player 1 (respectively, player 2) by female (respectively, male) pronouns and a generic player by plural pronouns. Let S_i and S_{-i} denote respectively the finite strategy sets of player i and that of their opponent. We denote the payoff to player i from choosing their strategy s_i in S_i, when their opponent has chosen s_{-i} in S_{-i} by $u_i(s_i, s_{-i})$. We shall model ambiguous beliefs of player i on S_{-i} with a particular *parsimoniously parameterized* class of capacities, the family of *neo-additive capacities*.

Definition 2.1 *A* capacity *on S_{-i} is a real-valued function ν_i on the subsets of S_{-i} such that $\nu_i(\varnothing) = 0$, $\nu_i(S_{-i}) = 1$, and, for each $A \subseteq B \subseteq S_{-i}$, $\nu_i(A) \leqslant \nu_i(B)$. Moreover, the capacity is a* neo-additive capacity *if there exists a probability distribution π_i on S_{-i} and $\alpha_i, \delta_i \in [0,1]$ such that for each non-empty (proper) subset $A \subset S_{-i}$, $\nu_i(A|\alpha_i, \delta_i, \pi_i) = \delta_i(1-\alpha_i) + (1-\delta_i)\pi_i(A)$.*[7]

The 'expected' payoff associated with a given strategy s_i in S_i of player i, with respect to the capacity ν_i on S_{-i} is taken to be the Choquet integral, defined as follows.

Definition 2.2 *The* Choquet integral *of $u_i(s_i, \cdot)$ with respect to the capacity ν_i on S_{-i} is:*

$$\begin{aligned} V_i(s_i|\nu_i) &= \int u_i(s_i, s_{-i})\, d\nu_i(s_{-i}) \\ &= u_i\left(s_i, s_{-i}^1\right)\nu_i\left(s_{-i}^1\right) + \sum_{r=2}^{R} u_i\left(s_i, s_{-i}^r\right)\left[\nu_i\left(s_{-i}^1, ..., s_{-i}^r\right) - \nu_i\left(s_{-i}^1, ..., s_{-i}^{r-1}\right)\right], \end{aligned}$$

where $R = |S_{-i}|$ and the strategy profiles in S_{-i} are numbered so that $u_i\left(s_i, s_{-i}^1\right) \geqslant u_i\left(s_i, s_{-i}^2\right) \geqslant ... \geqslant u_i\left(s_i, s_{-i}^R\right)$.

Fixing the two parameters α_i and δ_i, it is straightforward to show for any probability distribution π_i on S_{-i}, that the Choquet integral of $u_i(s_i, \cdot)$ with respect to the neo-additive capacity $\nu_i(\cdot|\alpha_i, \delta_i, \pi_i)$ on S_{-i} takes the simple and intuitive form of a weighted average between the expected utility with respect to π_i and the α-maxmin utility suggested by Hurwicz (1951) for choice under complete ignorance.

[7] As Proposition 3.1 (p541) in Chateauneuf, Eichberger, and Grant (2007) establishes, the parameters of a neo-additive capacity are only well-determined if there at at least three *essential* events, where an event is deemed essential if its capacity is neither zero nor one. Notice this will be satisfied if the opponent has at least three strategies.

Lemma 2.1 *The Choquet expected payoff with respect to the neo-additive capacity $\nu_i\left(\cdot|\alpha_i,\delta_i,\pi_i\right)$ on S_{-i} can be expressed as:*

$$V_i\left(s_i|\nu_i\left(\cdot|\alpha_i,\delta_i,\pi_i\right)\right)=\left(1-\delta_i\right)\cdot\mathbf{E}_{\pi_i}u_i\left(s_i,\cdot\right)+\delta_i\left[\alpha_i\min_{s_{-i}\in S_{-i}}u_i\left(s_i,s_{-i}\right)+\left(1-\alpha_i\right)\max_{s_{-i}\in S_{-i}}u_i\left(s_i,s_{-i}\right)\right], \tag{1}$$

where $\mathbf{E}_{\pi_i}u_i\left(s_i,\cdot\right)$ denotes a conventional expectation taken with respect to the probability distribution π_i on S_{-i}.[8]

Preferences that can be represented by a Choquet integral with respect to a neo-additive capacity are referred to as neo expected payoff maximizing.

One can interpret π_i as the "probabilistic belief" or "theory" of player i about how their opponent is choosing their strategy. However, players perceive there to be some degree of ambiguity associated with this theory. Their "confidence" in this theory is modelled by the weight $(1-\delta_i)$ given to the expected payoff $\mathbf{E}_{\pi_i}u_i\left(s_i,\cdot\right)$. Correspondingly, the highest (respectively, lowest) possible degree of ambiguity corresponds to $\delta_i=1$, (respectively, $\delta_i=0$). Their attitude toward such ambiguity is measured by α_i. Purely ambiguity-loving preferences are given by $\alpha_i=0$, while the highest level of ambiguity-aversion occurs when $\alpha_i=1$. If $0<\alpha_i<1$, then the player exhibits a mixed attitude toward ambiguity, since they respond to ambiguity partly in a pessimistic way by over-weighting the worst feasible payoff and partly in an optimistic way by over-weighting the best feasible payoff associated with that strategy.

2.2 The Support of a Capacity

We wish the support of a capacity to represent those strategies that a given player believes their opponent might play. Sarin and Wakker (1998) argue that the decision-maker's beliefs may be deduced from the decision weights in the Choquet integral. With this in mind, we propose the following definition.

Definition 2.3 *Fix ν_i a capacity on S_{-i}. Set*

$$\operatorname{supp}\left(\nu_i\right):=\left\{s_{-i}\in S_{-i}:\nu_i\left(A\cup s_{-i}\right)>\nu_i\left(A\right)\ \textit{for all}\ A\subset S_{-i}\backslash\left\{s_{-i}\right\}\right\}.$$

[8] To simplify notation we shall suppress the arguments and write $V_i\left(s_i|\nu_i\right)$ for $V_i\left(s_i|\nu_i\left(\cdot|\alpha_i,\delta_i,\pi_i\right)\right)$ when the meaning is clear from the context.

Notice that the set $\operatorname{supp}(\nu_i)$ consists of those strategies of i's opponent which always receive positive weight in the Choquet integral, no matter which of i's strategies is being evaluated. To see this, recall that the Choquet expected payoff of a given strategy s_i, is a weighted sum of pay-offs for which the decision-weight assigned to their opponent's strategy $\tilde{s}_{-i}$ is given by

$$\nu_i\left(\{s_{-i} : u_i(s_i, s_{-i}) > u_i(s_i, \tilde{s}_{-i})\} \cup \{\tilde{s}_{-i}\}\right) - \nu\left(\{s_{-i} : u_i(s_i, s_{-i}) > u_i(s_i, \tilde{s}_{-i})\}\right).$$

These weights depend on the way in which the strategy s_i ranks the strategies in S_{-i}. Since there are $R!$ ways the elements of S_{-i} can be ranked, in general there are $R!$ decision weights used in evaluating the Choquet integral with respect to a given capacity.

The next lemma shows that $\operatorname{supp}(\nu_i)$ yields an intuitive result when applied to neo-additive capacities.

Lemma 2.2 *Let ν_i be a neo-additive capacity on S_{-i} with $|S_{-i}| > 2$. Then* $\operatorname{supp}(\nu_i) = \operatorname{supp}\pi_i = \{s_{-i} \in S_{-i} : \pi_i(s_{-i}) > 0\}$.

Proof. Suppose $s_{-i} \notin A \subset S_{-i}$. If $A \cup \{s_{-i}\} \neq S_{-i}$, then $\nu_i(A \cup \{s_{-i}\}) - \nu_i(A)$

$= [\delta_i(1-\alpha_i) + (1-\delta_i)\pi_i(A \cup s_{-i})] - [\delta_i(1-\alpha_i) + (1-\delta_i)\pi_i(A)]$

$= [(1-\delta_i)\pi_i(A) + (1-\delta_i)\pi_i(s_{-i})] - [(1-\delta_i)\pi_i(A)] = (1-\delta_i)\pi_i(s_{-i})$. Alternatively, if $A \cup \{s_{-i}\} = S_{-i}$, then $\nu_i(A \cup \{s_{-i}\}) - \nu_i(A) = (1-\delta_i)\pi_i(s_{-i}) + \delta_i\alpha_i$. Thus $\nu_i(A \cup \{s_{-i}\}) > \nu_i(A)$ *for all* $A \subset S_{-i} \backslash \{s_{-i}\}$ if and only if $\pi_i(s_{-i}) > 0$. ■

Recall that we interpret π_i as the "probabilistic belief" or "theory" of an ambiguous belief. Thus it is natural that the support of the capacity ν_i is the support in the usual sense of the additive probability π_i.

2.3 Updating Ambiguous Beliefs

Neo-expected utility maximization is a theory of decision-making at one point in time. To use it in extensive form games we need to extend it to multiple time periods. We do this by employing Generalized Bayesian Updating (henceforth GBU) to revise beliefs. One problem which we face is that the resulting preferences may not be dynamically consistent. We respond to this by assuming that individuals take account of future preferences by using *consistent planning*, defined below. The GBU rule has been axiomatized in Eichberger, Grant, and Kelsey (2007) and Horie (2013). It is defined as follows.

Definition 2.4 *Let ν_i be a capacity on S_{-i} and let $E \subseteq S_{-i}$. The* Generalized Bayesian Update (henceforth GBU) *of ν_i conditional on E is given by:*

$$\nu_i^E(A) = \frac{\nu_i(A \cap E)}{\nu_i(A \cap E) + 1 - \nu_i(E^c \cup A)},$$

where $E^c = S_{-i} \backslash E$ denotes the complement of E.

The GBU rule coincides with Bayesian updating when beliefs are additive, for a neo-additive capacity we get the following.

Lemma 2.3 *For a neo-additive belief $\nu_i(\cdot|\alpha_i, \delta_i, \pi_i)$ the GBU conditional on E is given by*

$$\nu_i^E(A|\alpha_i, \delta_i, \pi_i) = \begin{cases} 0 & \text{if } A \cap E = \varnothing, \\ \delta_i^E(1-\alpha_i) + \left(1-\delta_i^E\right)\pi_i^E(A) & \text{if } \emptyset \subset A \cap E \subset E, \\ 1 & \text{if } A \cap E = E. \end{cases}$$

where $\delta_i^E = \delta_i / [\delta_i + (1-\delta_i)\pi_i(E)]$, and $\pi_i^E(A) = \pi_i(A \cap E)/\pi_i(E)$.

Notice that for a neo-additive belief with $\delta_i > 0$, the GBU update is well-defined even if $\pi_i(E) = 0$ (that is, E is a zero-probability event according to the individual's 'theory'). In this case the updated parameter $\delta_i^E = 1$, which implies the updated capacity is a Hurwicz capacity that assigns the weight 1 to every event that is a superset of E, and $(1-\alpha)$ to every event that is a non-empty strict subset of E.

As Eichberger, Grant, and Kelsey (2007) show a neo-expected payoff maximizer also admits a multiple priors representation. Pires (2002) shows the GBU update of a multiple prior model entails the prior by prior Bayesian update of the initial set of probabilities. But as the next result demonstrates there is a particularly tight connection between the GBU update of a neo-additive capacity and the multiple prior model. It states that a capacity is neo-additive, if *and only if* both it and its GBU update admit a multiple priors representation with the *same α_i parameter* and with the updated set of beliefs being comprised of the prior by prior Bayesian update of the initial set of probabilities.[9]

Proposition 2.1 *The capacity ν_i on S_{-i} is neo-additive for some parameters α_i and δ_i and some probability π_i, if and only if both the ex-ante and the updated preferences respectively admit multiple*

[9] A proof can be found in Eichberger, Grant, and Kelsey (2012).

priors representations of the form:

$$\int u_i(s_i, s_{-i})\, d\nu_i(s_{-i}) = \alpha_i \times \min_{q \in \mathcal{P}} \mathbf{E}_q u_i(s_i, \cdot) + (1 - \alpha_i) \times \max_{q \in \mathcal{P}} \mathbf{E}_q u_i(s_i, \cdot),$$

$$\int u_i(s_i, s_{-i})\, d\nu_i^E(s_{-i}) = \alpha_i \times \min_{q \in \mathcal{P}^E} \mathbf{E}_q u_i(s_i, \cdot) + (1 - \alpha_i) \times \max_{q \in \mathcal{P}^E} \mathbf{E}_q u_i(s_i, \cdot),$$

where $\mathcal{P} := \{p \in \Delta(S_{-i}) : p \geqslant (1 - \delta_i)\pi_i\}$, $\mathcal{P}^E := \left\{p \in \Delta(E) : p \geqslant \left(1 - \delta_i^E\right)\pi_i^E\right\}$, $\delta_i^E = \frac{\delta_i}{\delta_i + (1-\delta_i)\pi_i(E)}$, *and* $\pi_i^E(A) = \frac{\pi_i(A \cap E)}{\pi_i(E)}$.

We view this as a particularly attractive and intuitive result since the ambiguity-attitude, α_i, can be interpreted as a characteristic of the individual which is *not updated*. In contrast, the set of priors is related to the environment and one would expect it to be revised on the receipt of new information.[10]

2.4 Consistent Planning

As we have already foreshadowed, the combination of neo-expected payoff maximizing preferences and GBU updating is not, in general, dynamically consistent. Perceived ambiguity is usually greater after updating. Thus for an ambiguity-averse individual, constant acts will become more attractive. Hence if an individual is ambiguity-averse, in the future she may wish to take an option which gives a certain payoff, even if it was not in her original plan to do so. Following Strotz (1955), Siniscalchi (2011) argues against *commitment to a strategy* in a sequential decision problem in favor of *consistent planning.* This means that a player takes into account any changes in their own preference arising from updating at future nodes. As a result, players will take a sequence of moves which is consistent with backward induction. In general it will differ from the choice a player would make at the first move with commitment.[11] With consistent planning, however, dynamic consistency is no longer an issue.[12] The dynamic consistency issues and consistent planning are illustrated by the following example of individual choice in the presence of sequential resolution of uncertainty.

Example 2.1 *A decision maker (DM) is playing the following one-person two-stage game against nature. In the first stage the DM chooses between a safe action* s *or a risky action* r*. If the DM chooses* s *then the game ends and she gets a payoff of* $x \in (0,1)$*. If she chooses* r *then she observes*

[10] There are two alternative rules for updating ambiguous beliefs, the Dempster-Shafer (pessimistic) updating rule and the Optimistic updating rule, Gilboa and Schmeidler (1993). However neither of these will leave ambiguity-attitude, α_i, unchanged after updating. The updated α_i is always 1 (respectively, 0) for the Dempster-Shafer (respectively, Optimistic) updating rule. See Eichberger, Grant, and Kelsey (2010). For this reason we prefer the GBU rule.

[11] From this perspective, commitment devices should be explicitly modeled. If a commitment device exists, e.g., handing over the execution of a plan to a referee or writing an enforcible contract, then no future choice will be required.

[12] Bose and Renou (2014) and Karni and Safra (1989) use versions of consistent planning in games.

the first choice of nature: either ℓ (low) or h (high). If she observes ℓ then the game ends and she gets a payoff of 0. *If she observes h then in the second stage she again chooses between actions s or r. Similar to the first stage, if she chooses s in the second stage then the game ends and she gets a payoff of $y \in (0,1)$. If she chooses r then she observes nature's second choice which is either ℓ, yielding her a payoff of* 0, *or h, yielding her a payoff of* 1. *Let s, rs and rr denote the DM's three possible 'plans of actions' and ℓ, $h\ell$ and hh denote the three possible resolutions of uncertainty that the DM may observe. The extensive-form is illustrated below in figure 1 with squares (respectively, circles) denoting decision nodes for the DM (respectively, nature).*

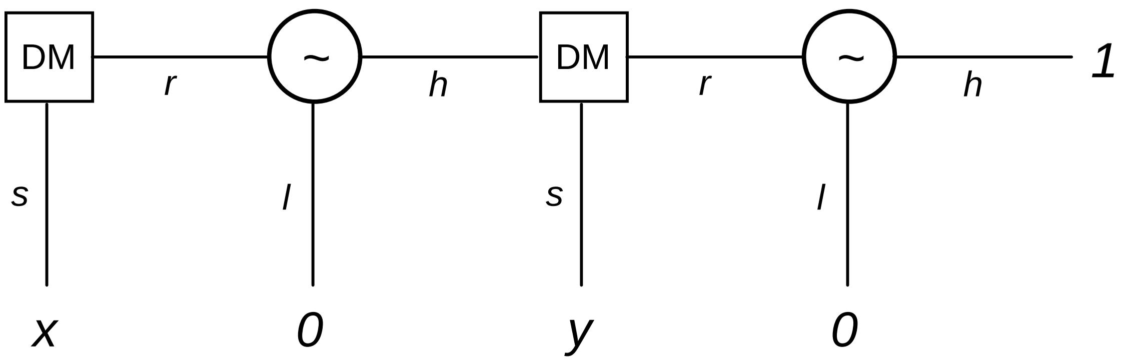

Figure 1: Sequential decision under uncertainty problem modeled as a 2-stage one-person game against nature.

Suppose that the individual is a neo-expected payoff maximizer. Her 'probabilistic belief' about nature's possible choices can be summarized by the pair of probabilities: $\pi(\ell) = 1 - p$ and $\pi(hh) = pq$, where $0 < \min\{p,q\}$ and $\max\{p,q\} < 1$. Her 'lack of confidence' in her belief is given by the parameter $\delta \in (0,1)$, and her attitude toward ambiguity is given by the parameter α which we take to be in the (open) interval $(1-q,1)$.

One-shot resolution. If the individual must commit in stage 1 to one of her three plans of action, then her choice is governed by her ex ante preferences represented by the neo-expected pay-offs:

$$V(s|\nu(\cdot|\alpha,\delta,\pi)) = x,\ V(rs|\nu(\cdot|\alpha,\delta,\pi)) = (1-\delta)py + \delta(1-\alpha)y$$

$$\text{and } V(rr|\nu(\cdot|\alpha,\delta,\pi)) = (1-\delta)pq + \delta(1-\alpha).$$

If y is set equal to q, then notice that $V(rr|\nu(\cdot|\alpha,\delta,\pi)) - V(rs|\nu(\cdot|\alpha,\delta,\pi)) = \delta(1-\alpha)(1-q) > 0$. Furthermore, if x is set so that $x = (1-\delta)pq + \delta(1-\alpha)(1+q)/2$, then we also have

$$V(s|\nu(\cdot|\alpha,\delta,\pi)) = \frac{1}{2}V(rs|\nu(\cdot|\alpha,\delta,\pi)) + \frac{1}{2}V(rr|\nu(\cdot|\alpha,\delta,\pi)).$$

So for a one-shot resolution scenario, the individual will strictly prefer to choose rr over both s and rs.

Sequential resolution. Now consider the scenario in which the individual has the opportunity to revise her plan of action in the second stage if nature's choice in the first stage is h. Let ν^h denote the GBU of ν conditional on nature having chosen h in the first stage. According to her *ex ante preferences*, she should stay with her plan of action rr. However, her choice of action in the second stage is governed by her *updated preferences* between rr and rs as represented by the pair of neo-expected pay-offs

$$V^h\left(rs|\nu^h\right) = q \quad \text{and} \quad V^h\left(rr|\nu^h\right) = \left(1-\delta^h\right)q + \delta^h(1-\alpha), \text{ where } \delta^h = \frac{\delta}{\delta + (1-\delta)p}.$$

Notice that $V^h\left(rs|\nu^h\right) - V^h\left(rr|\nu^h\right) = \frac{\delta}{\delta+(1-\delta)p} \times (\alpha - [1-q]) > 0$ which means we have established

$$V(rr|\nu) > V(rs|\nu) \text{ but } V^h\left(rs|\nu^h\right) > V^h\left(rr|\nu^h\right).$$

Hence rr does not constitute what we shall refer to in the sequel as a *consistent plan (of action)*, our notion of dynamically consistent choice.[13]

Notice that, if the individual were "naive" then in the sequential resolution setting, she would choose action r in the first stage, planning (in the event nature chooses h in stage 1) to choose action r again in the second stage. However, given her updated preferences, she would change her plan of action and choose the action s instead, yielding her a now guaranteed payoff of y $(= q)$. On the other hand, our consistent planner, anticipating her future self would not choose action r in the second stage, understands that her choice in the first stage is really between s and rs. Hence she selects s, since $V(s|\nu) > V(rs|\nu)$.

Remark 2.1 *Consistent Planning also has a behavioral component which psychologists relate to as*

[13] The formal statement of consistent planning appears in definition 3.3 in section 3.2.

the volitional control of emotions. Optimism and pessimism may be viewed as emotional responses to uncertainty which cannot be "quantified" by the frequency of observations. Being aware of such biases may stimulate "self control" and "self-regulation". Eisenberg, Smith, and Spinrad (2004, p. 263) write: "Effortful control pertains to the ability to wilfully or voluntarily inhibit, activate, or change (modulate) attention and behavior, as well as executive functioning tasks of planning, detecting errors, and integrating information relevant to selecting behavior." *Taking control of one's predictable biases, as it is suggested by consistent planning, is probably an essential property of human decision makings. Hence, consistent planning seems to be the adequate self-regulation strategy against dynamic inconsistencies in an uncertain environment for decision makers aware of their optimistic or pessimistic biases.*

3 Multi-stage Games of Almost Perfect Information

We turn now to a formal description of the sequential strategic interaction between two decision-makers. This is done by way of multi-stage games that have a fixed finite number of time periods. In any given period the history of previous moves is known to both players. Within a time period simultaneous moves are allowed. We believe these games are sufficiently general to cover many important economic applications entailing strategic interactions.

3.1 Description of the game.

There are 2 players, $i = 1, 2$ and T stages. At each stage t, $1 \leqslant t \leqslant T$, each player i simultaneously selects an action a_i^t.[14] Let $a^t = \left\langle a_1^t, a_2^t \right\rangle$ denote a profile of action choices by the players in stage t.

The game has a set $\overline{H}$ of histories h which:

1. contains the empty sequence $h^0 = \langle \varnothing \rangle$, (no records);
2. for any non-empty sequence $h = \left\langle a^1, ..., a^t \right\rangle \in \overline{H}$, all subsequences $\hat{h} = \left\langle a^1, ..., a^{\hat{t}} \right\rangle$ with $\hat{t} < t$ are also contained in $\overline{H}$.

The set of all histories at stage t are those sequences in $\overline{H}$ of length $t-1$, with the empty sequence h^0 being the only possible history at stage 1. Let H^{t-1} denote the set of possible histories at stage t with generic element $h^{t-1} = \left\langle a^1, ..., a^{t-1} \right\rangle$.[15] Any history $\left\langle a^1, ..., a^T \right\rangle \in \overline{H}$ of length T is a *terminal*

[14] It is without loss of generality to assume that each player moves in every time period. Games where one player does not move at a particular time, say $\hat{t}$, can be represented by assigning that player a singleton action set at time $\hat{t}$.

[15] Notice by definition, that $H^0 = \{h^0\}$.

history which we shall denote by z. We shall write Z $(= H^T)$ for the subset of $\overline{H}$ that are terminal histories. Let $H = \bigcup_{t=1}^{T} H^{t-1}$ denote the set of all non-terminal histories and let $\theta = |H|$ denote the number of non-terminal histories.[16] At stage t, all players know the history of moves from stages $\tau = 1$ to $t-1$.

For each $h \in H$ the set $A^h = \{a|\ (h,a) \in \bar{H}\}$ is called the action set at h. We assume that A^h is a Cartesian product $A^h = A_1^h \times A_2^h$, where A_i^h denotes the set of actions available to player i after history h. The action set, A_i^h, may depend both on the history and the player. A pure strategy specifies a player's move after every possible history.

Definition 3.1 *A* (pure) strategy *of a player* $i = 1,2$ *is a function* s_i *which assigns to each history* $h \in H$ *an action* $a_i \in A_i^h$*.*

Let S_i denote the strategy set of player i, $S = S_1 \times S_2$, the set of strategy profiles and $S_{-i} = S_j, j \neq i$, the set of strategies of i's opponent. Following the usual convention, we will sometimes express the strategy profile $s \in S$ as (s_i, s_{-i}), in order to emphasize that player i is choosing their strategy $s_i \in S_i$ given their opponent is choosing according to the strategy $s_{-i} \in S_{-i}$.

Each strategy profile $s = (s_1, s_2) \in S$ induces a sequence of histories $\langle h_s^1, \ldots, h_s^T \rangle$, given by $h_s^1 = \left(s_1\left(h^0\right), s_2\left(h^0\right)\right)$ and $h_s^t = \left\langle h_s^{t-1}, \left(s_1\left(h_s^{t-1}\right), s_2\left(h_s^{t-1}\right)\right)\right\rangle$, for $t = 2, \ldots, T$. This gives rise to a collection of functions $\left\langle \zeta^t \right\rangle_{t=0}^{T}$, where $\zeta^0(s) := h^0$ for every strategy profile $s \in S$ and for each $t = 1, \ldots, T$, the function $\zeta^t : S \to H^t$ is recursively constructed by setting $\zeta^t(s) := \left\langle \left(s_1(\zeta^{\tau-1}(s)), s_2(\zeta^{\tau-1}(s))\right)\right\rangle_{\tau=1}^{t}$. Each ζ^t is surjective since every history in H^t must arise from some combination of strategies.

A payoff function u_i for player i, assigns a real number to each terminal history $z \in Z$. With a slight abuse of notation, we shall write $u_i(s)$ for the convolution $u_i \circ \zeta^T(s)$. We now have all the elements to define a multi-stage game.

Definition 3.2 *A* multi-stage game Γ *is a triple* $\left\langle \{1,2\}, \overline{H}, \{u_1, u_2\} \right\rangle$, *where* $\overline{H}$ *is the set of all histories, and for* $i = 1,2$, u_i *characterizes player* i*'s pay-offs.*

3.2 Continuation Strategies, Conditional Pay-offs, and Consistent Planning.

A *(sub-) history* after a non-terminal history $h \in H$ is a sequence of actions h' such that $(h, h') \in \overline{H}$. Adopting the convention that (h, h^0) is identified with h, denote by $\overline{H}^h$ the set of histories following h. Let Z^h denote the set of terminal histories following h. That is, $Z^h = \left\{z' \in \overline{H}^h : (h, z') \in Z\right\}$.

[16] Notice that by construction $\bar{H} = H \cup Z$.

Consider a given individual, player i. Denote by s_i^h a (continuation-) strategy of player i which assigns to each history $h' \in \overline{H}^h \setminus Z^h$ an action $a_i \in A_i^{(h,h')}$. We will denote by S_i^h the set of all those (continuation-) strategies available to player i following the history $h \in H$ and define $S^h = S_1^h \times S_2^h$ to be the set of (continuation-) strategy profiles. Each strategy profile $s^h = \langle s_1^h, s_2^h \rangle \in S^h$ defines a terminal history in Z^h. Furthermore, we can take $u_i^h : Z^h \to \mathbb{R}$, to be the payoff function for player i given by $u_i^h(h') = u_i(h, h')$, and correspondingly set $u_i^h(s^h) := u_i^h(h')$ if the continuation strategy profile s^h leads to the play of the sub-history h'. Consider player i's choice of continuation strategy s_i^h in S_i^h that starts in stage t. To be able to compute their conditional neo-expected payoff, they first use Bayes' Rule to update their theory π_i (a probability measure defined on S_{-i}) to a probability measure defined on S_{-i}^h. In addition it is necessary to update their perception of ambiguity represented by the parameter δ_i. Now, since ζ^{t-1} is a surjection, there exists a well-defined pre-image $S(h) := \left(\zeta^{t-1}\right)^{-1}(h) \subseteq S$ for any history $h \in H^{t-1}$. The event $S_{-i}(h)$ is the marginal of this event on S_{-i} given by

$$S_{-i}(h) := \left\{ s_{-i} \in S_{-i} : \ \exists s_i \in S_i, \left(\zeta^{t-1}\right)^{-1}(s_i, s_{-i}) = h \right\}.$$

Similarly, the event $S_i(h)$ is the marginal of this event on S_i given by

$$S_i(h) := \left\{ s_i \in S_i : \ \exists s_{-i} \in S_{-i}, \left(\zeta^{t-1}\right)^{-1}(s_i, s_{-i}) = h \right\}.$$

Suppose that player i's initial belief about how the opponent is choosing a strategy is given by the neo-additive capacity $\nu_i = \nu_i(\alpha_i, \delta_i, \pi_i)$. Then, their evaluation of the neo-expected payoff associated with their continuation strategy s_i^h is given by:

$$V_i^h\left(s_i^h | \nu_i\right) = \left(1 - \delta_i^h\right) \mathbb{E}_{\pi_i^h} u_i\left(s_i^h, \cdot\right) + \delta_i^h \left[\alpha_i \min_{s_{-i}^h \in S_{-i}^h} u_i^h\left(s_i^h, s_{-i}^h\right) + (1 - \alpha_i) \max_{s_{-i}^h \in S_{-i}^h} u_i^h\left(s_i^h, s_{-i}^h\right) \right],$$

where $\delta_i^h = \delta_i / \left[\delta_i + (1 - \delta_i)\, \pi_i(S_{-i}(h))\right]$ (the GBU update of δ_i) and π_i^h is the Bayesian update of π_i whenever $\delta_i^h < 1$.

One-step deviations. Consider a given a history $h \in H^{t-1}$ and a strategy profile $s \in S$. A *one-step deviation* in stage t by player i from their strategy s_i to the action $a_i \in A_i^h$ leads to the terminal history in Z^h determined by the continuation strategy profile $\langle a_i, s_i^h(-t), s_{-i}^h \rangle$, where $s^h \in S^h$, is the continuation of the strategy profile s starting in stage t from history h, and $s_i^h(-t)$ is player

i's component of that strategy profile except for her choice of action in stage t. This enables us to separate player i's decision at stage t from the decisions of other players including their own past and future selves.

We now have all the elements needed to define what we mean for a strategy to be a *consistent plan of action*. Intuitively, we require that the player cannot increase their conditional neo-expected payoff by changing any single action in any single period.

Definition 3.3 *Fix a multi-stage game* $\langle \{1,2\}, \overline{H}, u_i, i = 1,2 \rangle$*. Given the (neo-additive) capacity* ν_i *on* S_{-i}*, the strategy* $s_i \in S_i$ *constitutes a* consistent plan *if*

$$V_i^h\left(s_i^h|\nu_i^h\right) \geqslant V_i^h\left(\left(a_i, s_i^h\left(-t\right)\right)|\nu_i^h\right),$$

for every $a_i \in A_i^h$*, every* $h \in H^{t-1}$*, and every* $t = 1, \ldots, T$*.*

4 Equilibrium Concept: Consistent Planning Equilibrium Under Ambiguity (CP-EUA)

In this section, we define an equilibrium notion which extends the concept of an Equilibrium under Ambiguity introduced in Eichberger and Kelsey (2014) to two-player multi-stage games where players choose a dynamically consistent plan at every stage of the game. Imposing consistent planning as part of the equilibrium condition is a major modification of the strategic-form equilibrium concept in Eichberger and Kelsey (2014). We will show that a CP-EUA exists for any attitude towards ambiguity $(\alpha_1, \alpha_2) \in [0,1]^2$ and all positive degrees of ambiguity $(\delta_1, \delta_2) \in (0,1]^2$. Moreover, with a mild condition on pay-offs, no weakly dominated strategy will be played in a CP-EUA.

We will also show that the equilibrium correspondence of a CP-EUA is upper hemi-continuous for all strictly positive degrees of ambiguity. In the limiting case of no ambiguity $(\delta_1, \delta_2) = (0,0)$, however, the neo-expected payoff of a continuation strategy s_i^h, $V_i^h\left(s_i^h|\nu_i\right)$, may be discontinuous. Hence, a limit of a sequence of CP-EUAs need not be a Nash equilibrium as ambiguity vanishes.

4.1 Existence

Our solution concept is an equilibrium in beliefs. Players choose pure (behavior) strategies, but have possibly ambiguous beliefs about the strategy choice of their opponent. Each player is required to choose at every decision node an action, which must be optimal with respect to their updated beliefs. When choosing an action a player treats his or her own future strategy as given. Consistency is achieved by requiring that the support of these beliefs is concentrated on the opponent's best replies. Thus it is a solution concept in the spirit of the agent normal form.

Definition 4.1 *Fix a multi-stage game* $\Gamma = \langle \{1,2\}, \overline{H}, u_i, i = 1,2 \rangle$. *A Consistent Planning Equilibrium Under Ambiguity (CP-EUA) is a profile of capacities* $\langle \nu_1, \nu_2 \rangle$ *such that for each player* $i = 1,2$, *every* $s_i \in \operatorname{supp} \nu_{-i}$ *is a consistent plan.*

Remark 4.1 *If* $|\operatorname{supp} \nu_i| = 1$ *for* $i = 1,2$ *we say that the equilibrium is* singleton*. Otherwise we say that it is* mixed*. Singleton equilibria are analogous to pure strategy Nash equilibria.*

CP-EUA requires that the continuation strategy that player i is planning to play from history $h \in H^{t-1}$ is in the support of their opponent's beliefs ν_{-i}. Moreover, the only strategies in the support of their opponent's beliefs ν_{-i} are ones in which the action choice at history $h \in H^t$ is optimal for player i given their updated capacity ν_i^h. This rules out "incredible threats" in dynamic games. Thus our solution concept is an ambiguous analogue of sub-game perfection.[17] Since we require beliefs to be in equilibrium in each subgame, an equilibrium at the initial node will imply optimal behavior of each player at each decision node.

Mixed equilibria should be interpreted as equilibria in beliefs. To illustrate this consider a given player (she). We assume that she chooses pure actions and any randomizing or uncertainty is in the mind of her opponent. We require beliefs to be consistent with actual behavior in the sense that pure strategies in the support of the beliefs induce behavior strategies, which are best responses at any node where the given player has the move. The combination of neo-expected payoff preferences and GBU updating is not, in general, dynamically consistent. A consequence of this is that in a mixed equilibrium some of the *pure strategies,* which the given player's opponents believe she may play, are not necessarily optimal at all decision nodes. This arises because her preferences may change when they are updated. In particular equilibrium pure strategies will typically not be indifferent at the initial node. However at any node she will choose *actions* which are best responses. Hence, any terminal history associated with any strategy profile $\langle s_1, s_2 \rangle$ in the set $\operatorname{supp} \nu_2 \times \operatorname{supp} \nu_1$ may arise as the *equilibrium play* in a CP-EUA.

[17] Recall that in a multi-stage game a new subgame starts after any given history h.

The following result establishes that when players are neo-expected payoff maximizers, an equilibrium exists for any exogeneously given degrees of ambiguity and ambiguity attitudes.

Proposition 4.1 *Let Γ be a multi-stage game with 2 neo-expected payoff maximizing players. Then Γ has at least one CP-EUA for any given parameters $\alpha_1, \alpha_2, \delta_1, \delta_2$, where $0 \leqslant \alpha_i \leqslant 1$, $0 < \delta_i \leqslant 1$, for $i = 1, 2$.*

4.2 Upper Hemi-Continuity

In this section we show that the CP-EUA gives rise to an equilibrium correspondence which is upper hemi continuous (uhc) in the parameters, α and δ. Recall uhc is defined as follows.

Definition 4.2 *Consider a set $A \subseteq \mathbb{R}^n$ and a compact set $B \subseteq \mathbb{R}^m$. A correspondence $\phi : A \to \mathcal{P}(B)$ is* upper hemi-continuous (uhc) *at a point $x^0 \in A$ if for any sequence $x^r \in A$ which converges to x^0 i.e., $\lim_{r\to\infty} x^r = x^0$, and any sequence $y^r \in \phi(x^r)$, $\lim_{r\to\infty} y^r = y^0$ implies $y^0 \in \phi(x^0)$.*

The following result establishes that the equilibrium correspondence is upper hemi-continuous in δ and α. This implies that an equilibrium will not suddenly disappear as δ or α changes, however new equilibria may appear.

Proposition 4.2 *Let $\Gamma = \left\langle \{1, 2\}, \overline{H}, u_i, i = 1, 2 \right\rangle$ be a 2-player multi-stage game. The correspondence $\Phi \langle \alpha_1, \alpha_2, \delta_1, \delta_2 \rangle$ defined by*

$$\Phi \langle \alpha_1, \alpha_2, \delta_1, \delta_2 \rangle := \left\{ (\pi_1, \pi_2) \in [0, 1]^2) | \, (\nu_1(\alpha_1, \delta_1, \pi_1), \nu_2(\alpha_2, \delta_2, \pi_2)) \in \mathcal{E} \right\},$$

where $\mathcal{E}$ denotes the set of CP-EUA, is upper hemi-continuous on $[0, 1]^2 \times (0, 1]^2$.

The proof of this proposition shows that a sequence of beliefs $\nu_i^n = \delta_i^n (1 - \alpha_i) + (1 - \delta_i^n) \pi_i^n$ converges to some $\overline{\nu}_i = \overline{\delta}_i (1 - \alpha_i) + \left(1 - \overline{\delta}_i\right) \overline{\pi}_i$ as the degrees of ambiguity converge $\delta_i^n \to \overline{\delta}_i > 0$

Moreover, it establishes that $\bar{s}_i \in \operatorname{supp} \overline{\pi}_i$ is a best response for Player i given beliefs $\bar{\nu}_i$. Consequently $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle$ is a CP-EUA of Γ.

4.3 Limits of CP-EUAs as ambiguity vanishes: $\delta \to 0$

Of particular interest is the limit of CP-EUAs as ambiguity vanishes, $\delta_i \to 0$ for $i = 1, 2$. Unfortunately, the neo-expected payoff of a continuation strategy s_i^h,

$$V_i^h\left(s_i^h|\nu_i\right) = \left(1-\delta_i^h\right)\mathbb{E}_{\pi_i^h} u_i\left(s_i^h,\cdot\right) + \delta_i^h\left[\alpha_i \min_{s_{-i}^h \in S_{-i}^h} u_i^h\left(s_i^h, s_{-i}^h\right) + (1-\alpha_i)\max_{s_{-i}^h \in S_{-i}^h} u_i^h\left(s_i^h, s_{-i}^h\right)\right], \tag{2}$$

where δ_i^h denotes the GBU update of δ_i,

$$\delta_i^h = \frac{\delta_i}{\left[\delta_i + (1-\delta_i)\,\pi_i\left(S_{-i}(h)\right)\right]}, \tag{3}$$

is discontinuous at $\delta_i = 0$, $\pi_i\left(S_{-i}(h)\right) = 0$. It is clear from Equation 3 that, for $\delta_i > 0$, δ_i^h converges to 1 as $\pi_i\left(S_{-i}(h)\right) \to 0$. For $\delta_i = 0$ however, $\delta_i^h = 0$ for any $\pi_i\left(S_{-i}(h)\right)$. Hence, for $\delta_i > 0$,

$$V_i^h\left(s_i^h|\nu_i\right) \to \left[\alpha_i \min_{s_{-i}^h \in S_{-i}^h} u_i^h\left(s_i^h, s_{-i}^h\right) + (1-\alpha_i)\max_{s_{-i}^h \in S_{-i}^h} u_i^h\left(s_i^h, s_{-i}^h\right)\right]$$

as $\pi_i\left(S_{-i}(h)\right) \to 0$, while, for $\delta_i = 0$, $V_i^h\left(s_i^h|\nu_i\right) = E_{\pi_i^h} u_i\left(s_i^h,\cdot\right)$.

A special feature of a CP-EUA is the fact that, in contrast to a Nash equilibrium, beliefs are well-defined at all information sets on and off the equilibrium path as long as there is some ambiguity, i.e., if $\delta = (\delta_1, \delta_2) \neq (0,0)$. If the equilibrium belief of a player $\pi_i^h\left(S_{-i}(h)\right)$ equals zero at some information set $S_{-i}(h)$, then Full Bayesian Updating implies that the degree of ambiguity will be updated to 1, $\delta_i^h = 1$, and the updated belief $\nu_i^h\left(\alpha_i, \delta_i^h, \pi_i^h\right)$ will be well-defined by the capacity $\nu_i^h\left(\alpha_i, 1, 0\right)\left(S_{-i}(h)\right) = (1-\alpha_i)$. For pure pessimism, $\alpha = 1$, one has $\nu_i^h\left(\alpha_i, 1, 0\right)\left(S_{-i}(h)\right) = 0$, which is the well-known *capacity of complete ignorance.*

This kind of updating appears quite intuitive in the spirit of traditional game-theoretic reasoning. A player who holds an equilibrium belief π_i about the opponent's behavior that implies that a certain information set will not be reached, but then unexpectedly finds himself at this information set, will be in deep doubt regarding the opponent's future behavior, as is reflected by the capacity of complete ignorance. At an information set off the equilibrium path where $\pi_i^h\left(S_{-i}(h)\right) = 0$, one can no longer argue that *"the degree of uncertainty is reduced through the assumption that each player knows the desires of the other players and the assumption that they will take whatever actions appear to gain their ends,"* Luce and Raiffa (1957, p. 275). In such a situation the ambiguity attitude, optimism or pessimism, will be the only guide for behavior.

A necessary condition for updating to complete ignorance in response to a failure of Bayesian updating is, however, a "grain of doubt", $\delta > 0$, regarding the strategic behavior of the opponent.

With full confidence Generalized Bayesian Updating will fail if some unforeseen move occurs. Hence, a sequence of CP-EUAs cannot approach a Nash equilibrium as ambiguity disappears.[18]

4.4 Dominance

In this section we show that under a mild assumption on pay-offs no weakly dominated strategies will be played in an CP-EUA. The intuition is as follows. If a player perceives his/her opponent's action to be ambiguous (s)he cannot rule out the possibility that a weakly dominated strategy may lead to a lower pay-off than the strategy which dominates it. Hence, weakly dominated strategies are eliminated provided a game's pay-offs have different extremes as defined below.

Definition 4.3 *Two strategies* $\hat{s}_i, \tilde{s}_i \in S_i$ *are said to have* different extremes *if* $m(\hat{s}_i) = m(\tilde{s}_i) \Rightarrow M(\hat{s}_i) \neq M(\tilde{s}_i)$, *where* $m(s_i) = \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$ *and* $M(s_i) = \max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$ *.A 2-player multi-stage game,* Γ *has* different extremes *if all pairs of strategies* $\hat{s}_i, \tilde{s}_i \in S_i$ *have different extremes for* $i = 1, 2$.

The following result shows that, in a game with different extremes, players will not use weakly dominated strategies in equilibrium.

Proposition 4.3 *Let* Γ *be a 2-player multi-stage game with different extremes. Then a weakly dominated strategy is not played in any CP-EUA of* Γ *for* $1 \geqslant \delta > 0$ *and* $0 < \alpha_i, \alpha_j < 1$.

5 The Centipede Game

In this section, we apply our analysis to the centipede game. This is a two-player game with perfect information. One aim is to see whether strategic ambiguity can contribute to explaining observed behavior. The centipede game was introduced by Rosenthal (1981) and studied in laboratory experiments by McKelvey and Palfrey (1992). A survey of subsequent experimental research can be found in Krokow, Colman, and Pulford (2016).

5.1 The Game

The centipede game may be described as follows. There are two people, player 1 (she) and player 2 (he). Between them is a table which contains $2M$ one-pound coins and a single two-pound coin. They

[18] Remark 5.1 provides an example in the context of the centipede game.

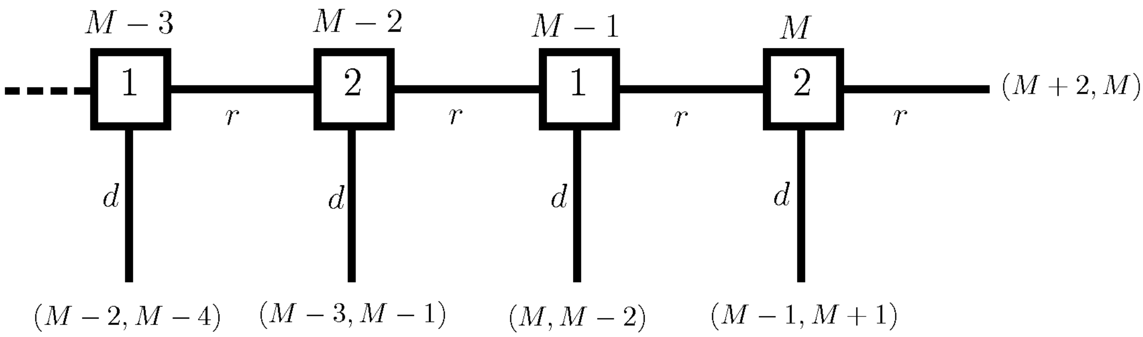

Figure 2: Last 4 stages of the Centipede Game

move alternately. At each move there are two actions available. The player whose move it is may either pick up the two-pound coin in which case the game ends; or (s)he may pick up two one-pound coins keep one and give the other to his/her opponent; in which case the game continues. In the final round there is a single two-pound coin and two one-pound coins remaining. Player 2, who has the move, may either pick up the two-pound coin, in which case the game ends and nobody gets the one-pound coins; or may pick up the two one-pound coins keep one and give the other to his opponent, in which case the opponent also gets the two pound coin and the game ends. We label an action that involves picking up two one-pound coins by r (right) and an action of picking up the two-pound coin by d (down). Figure 2 shows the final four decision nodes.

A standard backward induction argument establishes that there is a unique iterated dominance equilibrium. At any node the player, whose move it is, picks up the 2-pound coin and ends the game. There are other Nash equilibria. However these only differ from the iterated dominance equilibrium off the equilibrium path.

5.2 Notation

Given the special structure of the centipede game, we can simplify or notation. The tree can be identified with the non-terminal nodes $H = \{1, ..., M\}$. For simplicity we shall assume that M is an even number. The set of non-terminal nodes can be partitioned into the two player sets $H_1 = \{1, 3, ..., M-1\}$ and $H_2 = \{2, 4, ..., M\}$. It will be a maintained hypothesis that $M \geqslant 4$.

Strategies and Pay-offs. A (pure) strategy for player i is a mapping $s_i : H_i \to \{r, d\}$. Given a strategy combination (s_1, s_2) set $m(s_1, s_2) := 0$ if d is never played, otherwise set $m(s_1, s_2) := m' \in H$ where m' is the first node where action d is played. The payoff of strategy combination (s_1, s_2) is:

$u_1(s_1, s_2) = M + 2$, if $m(s_1, s_2) = 0; u_1(s_1, s_2) = m(s_1, s_2) + 1$ if $m(s_1, s_2)$ is odd and $u_1(s_1, s_2) = m(s_1, s_2) - 1$ otherwise. Likewise $u_2(s_1, s_2) = M$, if $m(s_1, s_2) = 0$; $u_2(s_1, s_2) = m(s_1, s_2) + 1$ if $m(s_1, s_2)$ is even; $u_2(s_1, s_2) = m(s_1, s_2) - 1$ otherwise. For each player $i = 1, 2$ and each node ρ in H_i, let s_i^ρ denote the threshold strategy defined as $s_i^\rho(m) = r$, for $m < \rho$; and $s_i^\rho(m) = d$ for $m \geqslant \rho$ Let s_i^∞ denote the strategy to play r always. Since threshold strategies $s_i^\rho, 1 \leqslant \rho \leqslant M - 1$, are weakly dominant they deserve special consideration. As we shall show, all equilibrium strategies are threshold strategies.

Subgames and Continuation Strategies. For any node $m' \in H$ set $m(s_1, s_2 \mid m') := 0$ if d is not played at m' or thereafter. Otherwise set $m(s_1, s_2 \mid m') := m'' \in \{m', ..., M\}$, where m'' is the first node where action d is played in the subgame starting at m'. We will write $u^i(s_1, s_2 \mid m')$ to denote the payoff of the continuation strategy from node m', that is, $u^1(s_1, s_2 \mid m') = M + 2$ if $m(s_1, s_2 \mid m') = 0$; $u_1(s_1, s_2 \mid m') = m(s_1, s_2 \mid m') + 1$ if $m(s_1, s_2 \mid m')$ is odd; $u_1(s_1, s_2 \mid m') = m(s_1, s_2 \mid m') - 1$ otherwise. Similarly define $u_2(s_1, s_2 \mid m') = M$ if $m(s_1, s_2 \mid m') = 0$; $u_2(s_1, s_2 \mid m') = m(s_1, s_2 \mid m') + 1$ if $m(s_1, s_2 \mid m')$ is even; $u_2(s_1, s_2 \mid m') = m(s_1, s_2 \mid m') - 1$ otherwise.

5.3 Consistent Planning Equilibria Under Ambiguity

In this section we characterize the CP-EUA of the centipede game with symmetric neo-expected payoff maximizing players. That is, throughout this section we take Γ to be an M stage centipede game, where M is an even number no less than 4, and in which both players are neo-expected payoff maximizers with $\delta_1 = \delta_2 = \delta \in [0, 1]$ and $\alpha_1 = \alpha_2 = \alpha \in [0, 1]$.

There are three possibilities, cooperation continues until the final node, there is no cooperation at any node or there is a mixed equilibrium. As we shall show below, a mixed equilibrium also involves a substantial amount of cooperation. The first proposition shows that if there is sufficient ambiguity and players are sufficiently optimistic the equilibrium involves playing "right" until the final node. At the final node Player 2 chooses "down" since it is a dominant strategy.

Proposition 5.1 *For $\delta(1 - \alpha) \geqslant \frac{1}{3}$, there exists a CP-EUA $\langle \nu_1(\cdot|\alpha, \delta, \pi_1), \nu_2(\cdot|\delta, \alpha, \pi_2) \rangle$, with $\pi_1(s_2^*) = \pi_2(s_1^*) = 1$ for the strategy profile $\langle s_1^*, s_2^* \rangle$ in which $m(s_1^*, s_2^*) = M$. This equilibrium will be unique provided the inequality is strict.*

This confirms our intuition. Ambiguity-loving preferences can lead to cooperation in the centipede game. To understand this result, observe that $\delta(1 - \alpha)$ is the decision-weight on the best outcome in the Choquet integral. Cooperation does not require highly ambiguity loving preferences. A necessary condition for cooperation is that ambiguity-aversion is not too high i.e. $\alpha \leqslant \frac{2}{3}$. Such ambiguity-attitudes are not implausible, since Kilka and Weber (2001) experimentally estimate that $\alpha = \frac{1}{2}$.

Recall that players do not cooperate in the Nash equilibrium. We would expect that ambiguity-aversion makes cooperation less likely, since it increases the attractiveness of playing down which offers a low but ambiguity-free payoff. The next result finds that, provided players are sufficiently ambiguity-averse, non-cooperation at every node is an equilibrium.

Proposition 5.2 *For $\alpha \geqslant \frac{2}{3}$, there exists a CP-EUA $\langle \nu_1 (\cdot|\alpha, \delta, \pi_1), \nu_2 (\cdot|\delta, \alpha, \pi_2)\rangle$, with $\pi_1 (s_2^*) = \pi_2 (s_1^*) = 1$ for the strategy profile $\langle s_1^*, s_2^* \rangle$, in which $m(s_1^*, s_2^*|m') = m'$ at every node $m' \in H$. This equilibrium will be unique provided the inequality is strict.*

It is perhaps worth emphasizing that pessimism must be large in order to induce players to exit at every node. If $\frac{1}{2} < \alpha < \frac{2}{3}$ the players overweight bad outcomes more than they overweight good outcomes. However non-cooperation at every node is not an equilibrium in this case even though players are fairly pessimistic about their opponents' behavior.

We proceed to study the equilibria when $\alpha < \frac{2}{3}$ and $\delta (1 - \alpha) < \frac{1}{3}$. This case is interesting, since Kilka and Weber (2001) estimate parameter values for α and δ in a neighborhood of $\frac{1}{2}$. The next result shows that there is no singleton equilibrium for these parameter values and characterizes the mixed equilibria which arise. Interestingly, the equilibrium strategies imply continuation for most nodes. This supports our hypothesis that ambiguity-loving can help to sustain cooperation.

Proposition 5.3 *Assume that $\delta (1 - \alpha) < \frac{1}{3}$ and $\alpha < \frac{2}{3}$. Then:*

1. *Γ does not have a singleton CP-EUA;*
2. *there exists a CP-EUA $\langle \nu_1 (\cdot|\alpha, \delta, \pi_1), \nu_2 (\cdot|\delta, \alpha, \pi_2)\rangle$ in which,*
   (a) *player 1 believes with degree of ambiguity δ that player 2 will choose his strategies with (ambiguous) probability $\pi_1(s_2) = p$ for $s_2 = s_2^M$; $1 - p$, for $s_2 = s_2^{M-2}$; $\pi_1(s_2) = 0$, otherwise, where $p = \frac{\delta(2-3\alpha)}{1-\delta}$;*
   (b) *player 2 believes with degree of ambiguity δ that player 1 will choose her strategies with (ambiguous) probability $\pi_2(s_1) = q$ for $s_1 = s_1^{M-1}$; $1 - q$, for $s_1 = s_1^{M-3}$; $\pi_2(s_1) = 0$, otherwise, where $q = \frac{1-3\delta(1-\alpha)}{3(1-\delta)}$;*

(c) The game will end at $M-2$ with player 2 exiting, at $M-1$ with player 1 exiting, or at M with player 2 exiting.

Notice that for the profile of admissible capacities $\langle \nu_1(\cdot|\alpha,\delta,\pi_1), \nu_2(\cdot|\delta,\alpha,\pi_2)\rangle$ specified in Proposition 5.3 to constitute a CP-EUA, we require player 2's "theory" about the "randomization" of player 1's choice of action at node $M-1$ should make player 2 at node $M-2$ indifferent between selecting either d or r. That is, $M-1 = (1-\delta)((1-q)(M-2)+q(M+1)) + \delta(\alpha(M-2)+(1-\alpha)(M+1))$, which solving for q yields,

$$q = \frac{1-3\delta(1-\alpha)}{3(1-\delta)}. \tag{4}$$

This is essentially the usual reasoning employed to determine the equilibrium 'mix' with standard expected payoff maximizing players.

The situation for player 1 is different, however, since her perception of the "randomization" undertaken by player 2 over his choice of action at node $M-2$ increases the ambiguity player 1 experiences at node $M-1$. Given full Bayesian updating, this should generate enough ambiguity for player 1 so that she is indifferent between her two actions at node $M-1$ given her "theory" that Player 2 will choose d at node M. More precisely, given the GBU of Player 1's belief conditional on reaching node $M-1$, player 1 should be indifferent between selecting either d or r; that is, $M = \left(1-\delta^{M-1}(1-\alpha)\right)(M-1) + \delta^{M-1}(M+2)$, where $\delta^{M-1} = \frac{\delta}{\delta+(1-\delta)p}$. Solving for p yields,

$$p = \frac{\delta(2-3\alpha)}{1-\delta}. \tag{5}$$

Thus substituting p into the expression above for δ^{M-1} we obtain $\delta^{M-1} = \frac{1}{3(1-\alpha)}$ and $\delta^{M-1}(1-\alpha) = \frac{1}{3}$, as required.

Remark 5.1 *It may at first seem puzzling that as $\delta \to 0$, we have $q \to \frac{1}{3}$, $p \to 0$ and $\delta^{M-1} = \frac{1}{3(1-\alpha)}$, for all $\delta \in \left(0, \frac{1}{3(1-\alpha)}\right)$, and yet for $\delta = 0$ (that is, with standard expected payoff maximizing players) by definition $\delta^{M-1} = 0$ and the unique equilibrium entails both players choosing d at every node, so in particular, $q = p = 0$. This discontinuity, is simply a consequence of the fact that (for fixed δ) $\frac{\delta}{\delta+(1-\delta)p} \to 1$ as $p \to 0$ in contrast to an intuition that the updated degree of ambiguity δ^{M-1} should converge to zero as $\delta \to 0$. Notice that for any (constant) $p > 0$, $\delta \to 0$ would indeed imply $\frac{\delta}{\delta+(1-\delta)p} \to 0$. However, to maintain an equilibrium of the type characterized in Proposition 5.3, p has to decrease sufficiently fast to maintain $\delta^{M-1} = \frac{1}{3(1-\alpha)}$.*

The discontinuity at $\delta = 0$ is puzzling if the intuition is guided by what one knows about mixed strategies and exogenous randomizations of pay-offs in perturbed games. Moreover this argues against interpreting any limit of a sequence of CP-EUA as $\delta \to 0$ as constituting a possible refinement of subgame perfect (Nash) equilibrium. Without optimism, there is no discontinuity, but then we are no longer able to explain the observed continuation in centipede games.[19]

The mixed equilibria occur when $\alpha < \frac{2}{3}$ and $\delta(1-\alpha) < \frac{1}{3}$. These parameter values could be described as situations of low ambiguity and low pessimism. On the equilibrium path players are not optimistic enough, given the low degrees of ambiguity, in order to play "right" at all nodes. However low pessimism makes them optimistic enough for playing "right" once they are off the equilibrium path whenever it is not a dominated strategy. This difference in behavior on and off the equilibrium path is the reason for non-existence of a singleton equilibrium.

In the mixed equilibrium the support of the original beliefs would contain two pure strategies, which player 1 has a strict preference between. However, at any node, where they differ, the behavior strategies which they induce are indifferent. (In these circumstances player 2 might well experience ambiguity concerning which strategy player 1 is following.)

The conditions $\alpha \gtreqqless \frac{2}{3}$ and $\delta(1-\alpha) \gtreqqless \frac{1}{3}$ characterize the parameter regions for the three types of *CP-EUA* equilibria. These are shown in figure 3. For strong pessimism ($\alpha > \frac{2}{3}$) players will always exit (red region), while for sufficient optimism and ambiguity ($\delta(1-\alpha) > \frac{1}{3}$) players will always continue (blue region). Kilka and Weber (2001) experimentally estimate the parameters of the neo-additive model as $\delta = \alpha = \frac{1}{2}$. For parameters in a neighborhood of these values only the mixed equilibrium exists. This would be compatible with a substantial degree of cooperation.

6 Bargaining

The alternating offer bargaining game, was developed by Stahl (1972) and Rubinstein (1982), has become one of the most intensely studied models in economics, both theoretically and experimentally. In its shortest version, the ultimatum game, it provides a prime example for a subgame perfect Nash equilibrium prediction at odds with experimental behavior. The theoretical prediction is of an initial offer of the smallest possible share of a surplus (often zero) followed by acceptance. However

[19] We thank stimulating comments and suggestions from David Levine and Larry Samuelson for motivating this remark.

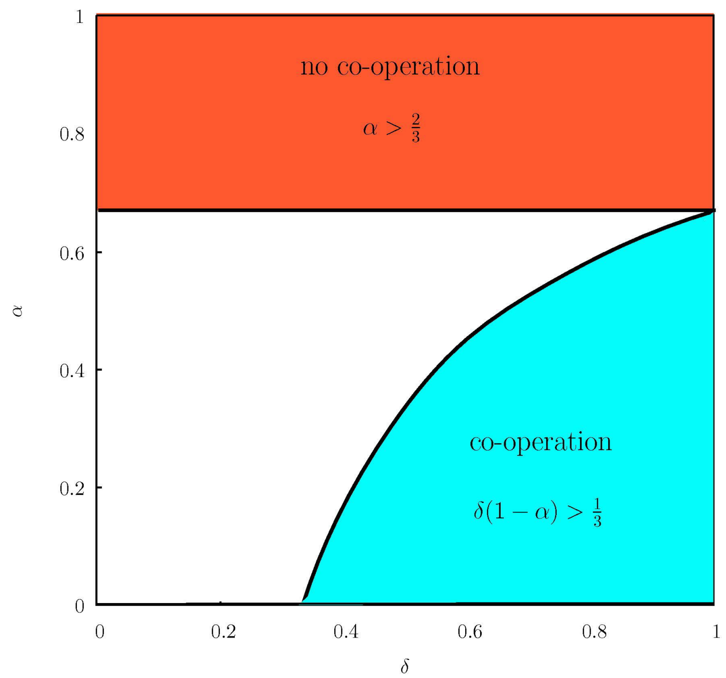

Figure 3: Centipede Game: Regions of the Parameter Space

experimental results show that the initial offers range around a third of the surplus which is often, but by far not always, accepted.

In bargaining games lasting for several rounds, the same subgame perfect equilibrium predicts a minimal offer depending on the discount rate and the length of the game, which will be accepted in the first round. Experimental studies show, however, that players not only make larger offers than suggested by the equilibrium but also do not accept an offer in the first round (Roth (1995), p. 293). In a game of perfect information rational agents should not waste resources by delaying agreement.

In order to accommodate the observed delays, game-theoretic analysis has suggested incomplete information about the opponent's pay-offs. Though it can be shown that incomplete information can lead players to reject an offer, the general objection to this explanation advanced in Forsythe, Kennan, and Sopher (1991) remains valid:

> "In a series of recent papers, the Roth group has shown that even if an experiment is designed so that each bargainer knows his opponent's utility pay-offs, the information structure is still incomplete. In fact, because we can never control the thoughts and beliefs of human subjects, it is impossible to run a complete information experiment. More generally, it is impossible to run an incomplete information experiment in which the experimenter knows the true information structure. Thus we must be willing to make conjectures about the beliefs which subjects might plausibly hold, and about how they may reasonably act in light of these beliefs. (p.243)"

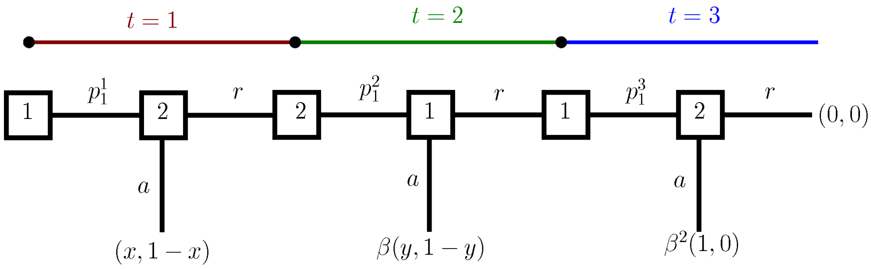

Figure 4: The bargaining game

In this paper we suggest another explanation. Following Luce and Raiffa (1957), p.275, we will assume that players view their opponent's behavior as ambiguous. Though this uncertainty will be reduced by their knowledge about the pay-offs of the other player and their assumption that opponents will maximize their payoff, players cannot be completely certain about their prediction. As we will show such ambiguity can lead to delayed acceptance of offers.

Consider the bargaining game in figure 4. Without ambiguity, backward induction predicts a split of $\langle \beta(1-\beta), 1-\beta(1-\beta) \rangle$ which will be accepted in period $t = 1$. Delay is not sensible because the best a player can expect from rejecting this offer is the same payoff (modulo the discount factor) a period later. Depending on the discount factor β the lion's share will go to the player who makes the offer in the last stage when the game turns into an ultimatum game.

Suppose now that a player feels some ambiguity about such equilibrium behavior. Since the incentives are balanced, it is plausible that a player might perceive ambiguity about his/her opponent's behavior. If a player has even a small degree of optimism, they may consider it possible that, by deviating from the expectations of the equilibrium path, the opponent may accept an offer which is more favorable for them. Hence, there may be an incentive to "test the water" by experimenting with a low offer. Note that this may be a low-cost deviation since, by returning to the previous path, just the discount is lost. Hence, if the discount is low, that is, the discount factor β is high, a small degree of optimism may suffice.

Decision makers with neo-expected payoff preferences who face ambiguity $\delta > 0$ and update their beliefs according to the GBU rule give some extra weight $(1-\alpha)$ to the best expected payoff and α to the worst expected payoff and update their beliefs to complete ambiguity, $\delta = 1$, if an event occurs which has probability zero according to their focal (additive) belief π. Hence, off the equilibrium path updates are well-defined but result in complete uncertainty. A decision maker with neo-additive beliefs will evaluate their strategies following an out of equilibrium move and, therefore probability zero event of π by their best and worst outcomes. Hence, from an optimistic perspective, asking for a high share may have a chance of being accepted resulting in some expected gain which can be balanced against the loss of discount associated with a rejection. Whether a strategy resulting in a delay is optimal will depend on the degree of ambiguity δ, the degree of optimism $1-\alpha$ and the discount factor β.

The following result supports this intuition. With ambiguity and some optimism, delayed agreement along the equilibrium path may occur in a CP-EUA equilibrium.

Proposition 6.1 *If* $\frac{\alpha-(1-\alpha)\beta}{1-(1-\alpha)[\max\{1-(1-\alpha)\beta,\beta\}+\beta]} \geqslant \delta$, *then there exists a CP-EUA* $\langle(\delta,\alpha,\pi_1),(\delta,\alpha,\pi_2)\rangle$ *such that* $\pi_1(s_2^*)=\pi_2(s_1^*)=1$ *for the following strategy profile* (s_1^*,s_2^*)*:*

- *at* $t=1$, *player 1 proposes division* $\langle x^*,1-x^*\rangle=\langle 1,0\rangle$, *player 2 accepts a proposed division* $\langle x,1-x\rangle$ *if and only if* $x\leqslant 1-[(1-\delta)(1-(1-\alpha)\beta)+\delta(1-\alpha)\beta\max\{1-(1-\alpha)\beta,\beta\}]$;
- *at* $t=2$,
  - *if player 1's proposed division in* $t=1$ *was* $\langle x,1-x\rangle=\langle 1,0\rangle$, *then player 2 proposes division* $\langle y^*,1-y^*\rangle=\langle(1-\alpha)\beta,1-(1-\alpha)\beta\rangle$ *and Player 1 accepts a proposed division* $\langle y,1-y\rangle$ *if and only if* $y\geqslant(1-\alpha)\beta$;
  - *otherwise, Player 2 proposes division* $\langle\hat{y},1-\hat{y}\rangle=\langle 0,1\rangle$, *Player 1 accepts the proposed division* $\langle y,1-y\rangle$ *if and only if* $y\geqslant(1-\alpha)\beta$;
- *at* $t=3$, *player 1 proposes division* $\langle z^*,1-z^*\rangle=\langle 1,0\rangle$ *and player 2 accepts any proposed division* $\langle z,1-z\rangle$.[20]

7 Relation to the Literature

This section relates the present paper to the existing literature. First we consider our own previous research followed by the relation to other theoretical research in the area. Finally we discuss the experimental evidence.

7.1 Ambiguity in Games

Most of our previous research has considered normal form games e.g. Eichberger and Kelsey (2014). The present paper extends this by expanding the class of games. Two earlier papers study a limited

[20] Note this is irrespective of Player 1's proposed division $\langle x,1-x\rangle$ at $t=1$, and irrespective of Player 2's proposed division $\langle y,1-y\rangle$ at $t=2$:

class of extensive form games, Eichberger and Kelsey (2004) and Eichberger and Kelsey (1999). These focus on signalling games in which each player only moves once. Consequently dynamic consistency is not a major problem. Signalling games may be seen as multi-stage games with only two stages and incomplete information. The present paper relaxes this restriction on the number of stages but has assumed complete information. The price of increasing the number of stages is that we are forced to consider dynamic consistency.

Rothe (2011) models each player's belief in an extensive form game as the special subclass of a neo-additive capacity with extreme pessimism (that is, $\alpha_i = 1$). However, he interprets the weight on the additive part of this capacity as the probability the player thinks her opponent is rational with the complementary weight being the probability she thinks her opponent is irrational. Furthermore, these (conditional) weights are specified exogeneously for each decision node of the player. It can thus be viewed as a Choquet expected utility extension of the irrational type literature discussed in section 8.1 below.

Hanany, Klibanoff, and Mukerji (2019) (henceforth HKM) also present a theory of ambiguity in multi-stage games. However they have made a number of different modelling choices. Firstly they consider games of incomplete information. In their model, there is ambiguity concerning the type of the opponent while their strategy is unambiguous. In contrast in our theory there is no type space and we focus on strategic ambiguity. However we believe that there is not a vast difference between strategic ambiguity and ambiguity over types. It would be straightforward to add a type space to our model, while HKM argue that strategic uncertainty can arise as a reduced form of a model with type uncertainty. Other differences are that HKM represent ambiguity by the smooth model, they use a different rule for updating beliefs and strengthen consistent planning to dynamic consistency. A cost of this is that they need to adopt a non-consequentialist decision rule. Thus current decisions may be affected by options which are no longer available. We conjecture that we could have obtained similar results using the smooth model. However, the GBU rule has the advantage that it defines beliefs both on and off the equilibrium path. In contrast, with the smooth rule, beliefs off the equilibrium path are to some extent arbitrary.

Jehiel (2005) proposes a solution concept which he refers to as *analogy-based equilibrium.* In this a player identifies similar situations and forms a single belief about their opponent's behavior in all of them. These beliefs are required to be correct in equilibrium. For instance in the centipede game a player might consider their opponent's behavior at all the non-terminal nodes to be analogous. Thus they may correctly believe that the opponent will play right with high probability at the average node, which increases their own incentive to play right. (The opponent perceives the situation similarly.) Jehiel predicts that either there is no cooperation or cooperation continues until the last decision node. This is not unlike our own predictions based on ambiguity.

What is common between his theory and ours is that there is an "averaging" over different decision nodes. In his theory this occurs through the perceived analogy classes, while in ours averaging occurs via the decision-weights in the Choquet integral. We believe that an advantage of our approach is that the preferences we consider have been derived axiomatically and hence are linked to a wider literature on decision theory.

7.2 Experimental Papers

Our paper predicts that ambiguity about the opponent's behavior may significantly increase cooperation above the Nash equilibrium level in the centipede game. This prediction is broadly confirmed by the available experimental evidence, (for a survey see Krokow, Colman, and Pulford (2016)).

McKelvey and Palfrey (1992) study 4 and 6-stage centipede games with exponential pay-offs. They find that most players play right until the last 3-4 stages, after which cooperation appears to break down randomly. This is compatible with our results on the centipede game which predict that cooperation continues until near the end of the game.[21]

Our paper makes the prediction that either their will be no cooperation in the centipede game or that cooperation will continue until the last three stages. In the latter case it will either break down randomly in a mixed equilibrium or break down at the final stage in a singleton equilibrium. In particular the paper predicts that cooperation will not break down in the middle of a long centipede game. This can in principle be experimentally tested. However we would note that it is not really possible to refute our predictions in a 4-stage centipede as used by McKelvey and Palfrey (1992) . Thus there is scope for further experimental research on longer games.[22]

[21] In an earlier draft we proved that there exist regions of the parameter space in which results analogous to Propositions 5.2 and 5.3 held for players with exponential utility.

[22] There are a number of other experimental papers on the centipede game. However many of them do not study the version of the game presented in this paper. For instance they may consider a constant sum centipede or study the normal form. It is not clear that our predictions will apply to these games. Because of this, we do not consider them in this review. For a survey see Krokow, Colman, and Pulford (2016).

8 Conclusion

This paper has studied extensive form games with ambiguity. This is done by constructing a thought experiment, where we introduce ambiguity but otherwise make as few changes to standard models as possible. We have proposed a solution concept for multi-stage games with ambiguity. An implication of this is that singleton equilibria may not exist in games of complete and perfect information. This is demonstrated by the fact that the centipede game only has mixed equilibria for some parameter values. We have shown that ambiguity-loving behavior may explain apparently counter-intuitive properties of the (unique) subgame perfect equilibrium in the Centipede game and non-cooperative bargaining. It also produces predictions closer to the available evidence than the subgame perfect equilibrium.

8.1 Irrational Types

As mentioned in the introduction, economists have been puzzled about the deviations from subgame perfect equilibrium predictions in a number of games such as the centipede game, the repeated prisoners' dilemma and the chain store paradox. In the present paper we have attempted to explain this behavior as a response to ambiguity. Previously it has been common to explain these deviations by the introduction of an "irrational type" of a player. This converts the original game into a game of incomplete information where players take into consideration a small probability of meeting an irrational opponent. An "irrational" player is a type whose pay-offs differ from the corresponding player's pay-offs in the original game. In such modified games of incomplete information, it can be shown that the optimal strategy of a "rational" player may involve imitating the "irrational" player in order to induce more favorable behavior by his/her opponents. This method is used to rationalize observed behavior in the repeated prisoner's dilemma, chainstore paradox (Kreps, Milgrom, Roberts, and Wilson (1982)), and in the centipede game (McKelvey and Palfrey (1992)).

There are at least two reasons why resolving the conflict between backward induction and observed behavior by introducing "irrational" players may not be the complete answer. First, games of incomplete information with "irrational" players predict with small probabilities that two irrational types will confront each other. Hence, this should appear in the experimental data. Secondly, in order to introduce the appropriate "irrational" types, one needs to know the observed deviations from equilibrium behavior. Almost any type of behavior can be justified as a response to some kind of irrational opponent. It is plausible that one's opponent may play tit for tat in the repeated prisoners' dilemma. Thus an intuitive account of cooperation in the repeated prisoners' dilemma may be based on a small probability of facing an opponent of this type. However, for most games, there is no such focal strategy which one can postulate for an irrational type to adopt. Theory does not help to determine which irrational types should be considered and hence does not make usually clear predictions. In contrast our approach is based on axiomatic decision theory and can be applied to any two-player multi-stage game.

8.2 Directions for Future Research

In the present paper we have focused on two-player multi-stage games. There appears to be scope for extending our analysis to a larger class of games. For instance, we believe that it would be straightforward to add incomplete information by including a type space for each player. Extensions to games with more than two players are possible. If there are three or more players, however, it is usual to assume that each one believes that their opponents act independently and at present it is not clear as to how one should best model independence of ambiguous beliefs.[23]

It should also be possible to extend the results to a larger class of preferences. Our approach is suitable for any ambiguity model which maintains a separation between beliefs and tastes and allows a suitable support notion to be defined. In particular both the multiple priors and smooth models of ambiguity fit these criteria. These models represent beliefs by a set of probabilities. A suitable support notion can be defined in terms of the intersection of the supports of the probabilities in this set of beliefs. This is the inner support of Ryan (2002).

A natural application is to finitely repeated games. Such games have some features in common with the centipede game. If there is a unique Nash equilibrium of the stage game then backward induction implies that there is no scope for cooperation in the repeated game. However in examples, such as the repeated prisoners' dilemma, intuition suggests that some cooperation should be possible.

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[23] There are still some differences of opinion among the authors of this paper on this point.

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A Appendix: Proofs

A.1 Existence of Equilibrium

In this sub-appendix we present the proof of the existence result, Proposition 4.1.

The strategy of our proof is to associate with Γ a modified game Γ', which is based on the agent-normal form of Γ. We show that Γ' has a Nash equilibrium and then use this to construct a CP-EUA for Γ. The game Γ' has 2θ players.[24] A typical player is denoted by $i_{h(t)}$, $h(t) \in H \backslash Z, i = 1, 2$. Thus there are 2 players for each decision node in Γ.

The strategy set of Player $i_{h(t)}, \Sigma_{i_{h(t)}} = \Delta\left(A_i^{h(t)}\right)$ is the set of all probability distributions over $A_i^{h(t)}$, with generic element $s_{i_{h(t)}} \in \Sigma_{i_{h(t)}}$, for $i = 1, 2$. Hence in game Γ', Player $i_{h(t)}$ may choose any mixed strategy over the set of actions $A_i^{h(t)}$. Let $\pi(h(t), \rho)$ denote the probability of history $h(t)$ when the strategy profile is ρ. This is calculated according to the usual rules for reducing compound lotteries to simple lotteries. We shall suppress the arguments and write $\pi = \pi(h(t), \rho)$ when convenient. Let π_i denote the marginal of π on S_{-i}.

The payoff of Player $i_{h(t)}$ is $\phi_{i_{h(t)}} : \Sigma_{i_{h(t)}} \to \mathbb{R}$, defined by $\phi_{i_{h(t)}}\left(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t\right)$ $= \int u_i\left(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t\right) d\nu_{i_{h(t)}}$, where ν_i is the neo-additive capacity on S_{-i} defined by $\nu_i(\varnothing) = 0; \nu_i(A) = \delta_i(1-\alpha_i)\pi_i(A), \varnothing \subsetneqq A \subsetneqq S_{-i}; \nu_i(S_{-i}) = 1$ and $\nu_{i_{h(t)}}$ is the GBU update of ν_i conditional on $h(t)$. Since $\nu_{i_{h(t)}}$ is neo-additive:

$$\begin{aligned}\phi_{i_{h(t)}}\left(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t\right) &= \delta_{i_{h(t)}}(1-\alpha_i) M_{i_{h(t)}}\left(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t\right) + \delta_{i_{h(t)}}\alpha_i m_{i_{h(t)}}\left(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t\right) \\ &\quad + \left(1-\delta_{i_{h(t)}}\right)\mathbf{E}_{\pi^i_{h(t)}} u_i\left(h(t), a_i^{h(t)}, s_i^{t+1}, s_{-i}^t\right), \end{aligned} \tag{6}$$

where $M_{i_{h(t)}}\left(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t\right) = \max_{s_{-i}^t \in S_{-i}^t} u_i\left\langle h^t, a_i^{h(t)}, s_i^{t+1}, s_{-i}^t\right\rangle$, and $m_{i_{h(t)}}\left\langle a_i^{h(t)}, s_i^{t+1}, s_{-i}^t\right\rangle$ $= \min_{s_{-i}^t \in S_{-i}^t} u_i\left\langle h^t, a_i^{h(t)}, s_{-i}^t\right\rangle$. Here $\pi_i^{h(t)}$ denotes the Bayesian update of π^i, given that node $h(t)$ has been reached and $h(t)$ has positive probability. (If $h(t)$ has probability 0, then $\delta_{i_{h(t)}} = 1$ and $\pi^i_{h(t)}$ can be any probability distribution over S_{-i}^t.)

If Player $i_{h(t)}$ plays a mixed strategy then his/her action may be described by a probability distribution ρ over $A_i^{h(t)}$, which is treated as an *ex-ante* randomization. Eichberger, Grant, and Kelsey

[24] Recall $\theta = |H \backslash Z|$ denotes the number of non-terminal histories.

(2016) show that individuals will be indifferent to ex-ante randomizations. Hence it is evaluated as

$$\sum_{a \in A_i^{h(t)}} \rho(a) \, \phi_{i_{h(t)}} \left(a, s_i^{t+1}, s_{-i}^t\right). \tag{7}$$

It follows that $i_{h(t)}$'s preferences are linear and hence quasi-concave in his/her own strategy.[25]

Likewise if one of $i_{h(t)}$'s "future selves" randomizes this is evaluated as $\sum_{s_i^{t+1} \in S_i^{t+1}} \xi(a) \, \phi_{i_{h(t)}} \left(a, s_i^{t+1}, s_{-i}^t\right)$, where ξ is the probability distribution over S_i^{t+1} induced by future randomizations. This is treated as an ex-ante randomization because it is resolved before the strategic ambiguity arising from the choice of i's opponent in the relevant subgame. We do not need to specify $i_{h(t)}$'s reaction to randomizations by his/her past selves since these are, by definition, already resolved at the point where the decision is made.

Lemma A.1 *The function $\phi_{i_{h(t)}} \left(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t\right)$ is continuous in s, provided $1 \geqslant \delta_i > 0$, for $i = 1, 2$.*

Proof. Consider equation (6). First note that $\phi_{i_{h(t)}}$ depends directly on s via the $\left(a_i^{h(t)}, s_i^{t+1}, s_{-i}^t\right)$ term. It also depends indirectly on s since the degree of ambiguity $\delta_{i_{h(t)}}$ and $\pi_{i_{h(t)}}$ are functions of s.

It follows from our assumptions that the direct relation between s and ϕ is continuous. By equation (7), ϕ is continuous in $\pi_{i_{h(t)}}$. Thus we only need to consider whether $\delta_{i_{h(t)}}$ and $\pi_{i_{h(t)}}$ are continuous in s. Recall that $\pi_{i_{h(t)}}$ is the probability distribution over terminal nodes induced by the continuation strategies s_i^t, s_{-i}^t. Since this is obtained by applying the law of compound lotteries it depends continuously on s. By definition $\delta_i^h = \frac{\delta_i}{\delta_i + (1-\delta_i)\pi_i(S_{-i}(h))}$. This is continuous in $\pi(h(t))$ provided the denominator is not zero, which is ensured by the condition $\delta_i > 0$. Since $\pi(h(t))$ is a continuous function of s, the result follows. ■

The next result establishes that the associated game Γ' has a standard Nash equilibrium.

Lemma A.2 *The associated game Γ' has a Nash equilibrium provided $1 \geqslant \delta_i > 0$, for $i = 1, 2$.*

Proof. In the associated game Γ', the strategy set of a typical player, $i_{h(t)}$, is the set of all probability distributions over the finite set $A_i^{h(t)}$ and is thus compact and convex. By equation (7) the payoff, $\phi_{i_{h(t)}}$, of Player $i_{h(t)}$ is continuous in the strategy profile σ. Moreover $\phi_{i_{h(t)}}$ is quasi concave in own strategy by equation (7). It follows that Γ' satisfies the conditions of Nash's theorem and therefore has a Nash equilibrium in mixed strategies. ■

[25] To clarify these remarks about randomization apply to the modified game Γ'. In the original game Γ there is an equilibrium in beliefs and no randomization is used.

Proposition 4.1 *Let Γ be a 2-player multi-stage game. Then Γ has at least one CP-EUA for any given parameters $\alpha_1, \alpha_2, \delta_1, \delta_2$, where $1 \leqslant \alpha_i \leqslant 0, 0 < \delta_i \leqslant 1$, for $i = 1, 2$.*

Proof. Let $\rho = \left\langle \rho^{i_{h(t)}} : i = 1, 2, h \in H \backslash Z \right\rangle$ denote a Nash equilibrium of Γ'. We shall construct a CP-EUA $\hat{\sigma}$ of Γ based on ρ. Note that ρ may be viewed as a profile of behavior strategies in Γ. Let $\hat{s}$ denote the profile of mixed strategies in Γ, which corresponds to ρ, and let π denote the probability distribution which ρ induces over S. (If ρ is an equilibrium in pure strategies then π will be degenerate.) The beliefs of player i in profile $\hat{\sigma}$ are represented by a neo-additive capacity $\hat{\nu}_i$ on S_{-i}, defined by $\hat{\nu}_i (B) = \delta_i (1 - \alpha_i) + (1 - \delta_i) \pi_i (B)$, where $B \subseteq S_{-i}$ and π_i denotes the marginal of π on S_{-i}.

Let $\hat{\nu}$ denote the profile of beliefs $\hat{\nu} = \langle \hat{\nu}_1, \hat{\nu}_2 \rangle$. We assert that $\hat{\nu}$ is a CP-EUA of Γ. By Subsection 3.2, it is sufficient to show that no player can increase his or her current utility by a one-step deviation. Consider a typical player j. Let $\hat{t}$, $0 \leqslant \hat{t} \leqslant T$, be an arbitrary time period and consider a given history $\hat{h}\left(\hat{t}\right)$ at time $\hat{t}$. Let $\bar{a}_j^{\hat{t}} \in A_{-j}^{h(\hat{t})}$ denote an arbitrary action for j at history $\hat{h}\left(\hat{t}\right)$. Since ρ is an equilibrium of Γ',

$$\phi_{j_{\hat{h}(\hat{t})}} \left(\tilde{a}_j^{\hat{t}}, s_j^{\hat{t}+1}, s_{-j}^{\hat{t}} \right) \geqslant \phi_{j_{\hat{h}(\hat{t})}} \left(\bar{a}_j^{\hat{t}}, s_j^{\hat{t}+1}, s_{-j}^{\hat{t}} \right),$$

for any $\tilde{a}_j^{\hat{t}} \in \operatorname{supp} \rho(j) = \operatorname{supp} \hat{\nu}_j^{\hat{h}(\hat{t})}$, where $\rho(j)$ denotes the marginal of ρ on $A_j^{h(\hat{t})}$. Without loss of generality we may assume that $\tilde{a}_j^{\hat{t}} = \hat{s}_j^{\hat{h}(\hat{t})} \in \operatorname{supp} \hat{\nu}_j$. By definition $\phi_{j_{\hat{h}(\hat{t})}} \left(\tilde{a}_j^{\hat{t}}, s_j^{\hat{t}+1}, s_{-j}^{\hat{t}} \right) = \int u_j \left(\hat{s}_j^{\hat{h}(\hat{t})}, \hat{s}_{\langle j,1 \rangle}^{t+1}, s_{-j}^{t} \right) d\nu_j^{\hat{h}(\hat{t})}$, where $\nu_j^{\hat{h}(\hat{t})}$ is the GBU update of ν_j conditional on $\hat{h}\left(\hat{t}\right)$. Since the behavior strategy $\hat{s}_j^{\hat{h}(\hat{t})}$ is by construction a best response at $\hat{h}\left(\hat{t}\right)$, $\int u_j \left(\hat{s}_j^{\hat{h}(\hat{t})}, \hat{s}_j^{t+1}, s_{-j}^{t} \right) d\nu_j^{\hat{h}(\hat{t})}$ $\geqslant \int u_j \left(\bar{a}_j^{\hat{t}}, \hat{s}_j^{t+1}, s_{-j}^{t} \right) d\nu_j^{\hat{h}(\hat{t})}$, which establishes that $\hat{s}_j^{\hat{h}(\hat{t})}$ yields a higher payoff than the one step deviation to $\bar{a}_j^{\hat{t}}$. Since both j and $\hat{h}\left(\hat{t}\right)$ were chosen arbitrarily this establishes that it is not possible to improve upon $\hat{\sigma}$ by a one-step deviation. Hence $\hat{\sigma}$ is a CP-EUA of Γ. ■

A.1.1 Upper hemi-continuity

Proof of Proposition 4.2 For all $\langle \alpha_1, \alpha_2, \delta_1, \delta_2 \rangle \in [0,1] \times (0,1]^2$, by Proposition 4.1, $\Phi \langle \alpha_1, \alpha_2, \delta_1, \delta_2 \rangle$ is not empty. Consider a sequence of parameter values $\langle \alpha_1^r, \alpha_2^r, \delta_1^r, \delta_2^r \rangle$ such that $\lim_{r \to \infty} \alpha_1^r = \bar{\alpha}_1$, $\lim_{r \to \infty} \alpha_2^r = \bar{\alpha}_2, \lim_{r \to \infty} \delta_1^r = \bar{\delta}_1$ and $\lim_{r \to \infty} \delta_2^r = \bar{\delta}_2$. Let $\langle \nu_1^r, \nu_2^r \rangle$ be a CP-EUA when the parameter values are $\langle \alpha_1^r, \alpha_2^r, \delta_1^r, \delta_2^r \rangle$. Since ν_1^r and ν_2^r are neo-additive we may write $\nu_i^r = \delta_i^r (1 - \alpha_i^r) + (1 - \delta_i^r) \pi_i^r (A)$, for $i = 1, 2$.

By taking a subsequence, if necessary, we may assume that $\langle \pi_1^r, \pi_2^r \rangle \in \Phi(\alpha_1^r, \alpha_2^r, \delta_1^r, \delta_2^r)$ converges to a limit $\langle \bar{\pi}_1, \bar{\pi}_2 \rangle$. Define a neo-additive capacity, $\bar{\nu}_i$, by $\bar{\nu}_i = \bar{\delta}_i (1 - \bar{\alpha}_i) + (1 - \bar{\delta}_i) \bar{\pi}_i (A)$, for $i = 1, 2$.

Since $\langle \nu_1^r, \nu_2^r \rangle$ is a CP-EUA, if $\tilde{s}_i \in \operatorname{supp} \nu_j^r = \operatorname{supp} \pi_j^r$,

$$V_i^h \left(\tilde{s}_i^h | \nu_i^h \left(\alpha_i^r, \delta_i^r, \pi_i^r \right) \right) \geqslant V_i^h \left(a_i^r, \tilde{s}_i^h \left(-t \right) | \nu_i^h \left(\alpha_i^r, \delta_i^r, \pi_i^r \right) \right), \tag{8}$$

for $i, j = 1, 2$, for every $a_i \in A_i^h$, every $h \in H^{t-1}$, and every $t = 1, \ldots, T$.

Consider $\bar{s}_1 \in \operatorname{supp} \bar{\pi}_1$. Then there exists a sequence $s_1^r \in \operatorname{supp} \pi_1^r$ such that $\lim_{r \to \infty} s_1^r = \bar{s}_1$. Suppose, if possible, there exists $a_1 \in A_1^h$, some t and some $h \in H^{t-1}$ such that

$$V_1^h \left(\bar{s}_1^h | \nu_1^h \left(\alpha_1^r, \bar{\delta}_1, \bar{\pi}_1 \right) \right) < V_1^h \left(a_1^r, \bar{s}_1^h \left(-t \right) | \nu_1^h \left(\alpha_1^r, \bar{\delta}_1, \bar{\pi}_1 \right) \right). \tag{9}$$

By continuity of V_1^h in δ_1 and π_1, there must be some N such that

$$V_1^h \left(\tilde{s}_1^{hr} | \nu_1^h \left(\alpha_1^r, \delta_1^r, \pi_1^r \right) \right) < V_1^h \left(a_1^r, \tilde{s}_1^{hr} \left(-t \right) | \nu_1^h \left(\alpha_1^r, \delta_1^r, \pi_1^r \right) \right),$$

for all $r > N$, contradicting equation (8).

This establishes that $\bar{s}_1 \in \operatorname{supp} \bar{\pi}_1$ is a best response for Player 1 given beliefs $\bar{\nu}_1$. A similar argument establishes that if $\bar{s}_2 \in \operatorname{supp} \bar{\pi}_2$, then $\bar{s}_2$ is a best response for Player 2, given beliefs $\bar{\nu}_2$ and consequently that $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle$ is a CP-EUA of Γ. ■

A.1.2 Dominance

Proof of Proposition 4.3 Let $\langle \hat{\nu}_1, \hat{\nu}_2 \rangle$ be a CP-EUA of Γ, where $\hat{\nu}_i = \delta_i (1 - \alpha_i) \hat{\pi}_i$, and $\hat{\pi}_i$ is an additive probability for $i = 1, 2$. Suppose, if possible, that some $\hat{s}_i$ in the support of the equilibrium beliefs is weakly dominated by strategy $\tilde{s}_i$, i.e., $u_i (\tilde{s}_i, s_{-i}) \geqslant u_i (\hat{s}_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ with strict inequality for some s_{-i}. Hence

$$\mathbf{E}_{\hat{\pi}_{-i}} u_i (\tilde{s}_i, s_{-i}) \geqslant \mathbf{E}_{\hat{\pi}_{-i}} u_i (\hat{s}_i, s_{-i}).$$

Since $\tilde{s}_i$ weakly dominates $\hat{s}_i$ $m_i(\tilde{s}_i) \geqslant m_i(\hat{s}_i)$ and $M_i(\tilde{s}_i) \geqslant M_i(\hat{s}_i)$. Moreover the different extremes assumption tells us that if $m_i(\tilde{s}_i) = m_i(\hat{s}_i)$ then $M_i(\tilde{s}_i) > M_i(\hat{s}_i)$. Therefore

$$\begin{aligned} & V_i(\tilde{s}_i) = \delta_i(1-\alpha_i) M_i(\tilde{s}_i) + \delta_i \alpha_i m_i(\tilde{s}_i) + (1-\delta_i) \mathbf{E}_{\hat{\pi}_{-i}} u_i(\tilde{s}_i, s_{-i}) \\ > \ & \delta_i(1-\alpha_i) M_i(\hat{s}_i) + \delta_i \alpha_i m_i(\hat{s}_i) + (1-\delta_i) \mathbf{E}_{\hat{\pi}_{-i}} u_i(\hat{s}_i, s_{-i}) = V_i(\hat{s}_i), \end{aligned}$$

which shows that $\hat{s}_i$ cannot be a consistent plan. ■

A.2 The Centipede Game

Proof of Proposition 5.1 We shall proceed by (backward) induction. The final node is M. At this node 2 plays d_M, which is a dominant strategy. This yields pay-offs $\langle M-1, M+1\rangle$.

Node $M-1$ Now consider Player 1's decision at node $M-1$. Assume that this node is on the equilibrium path.[26] The (Choquet) expected value of her pay-offs are:

$V_1^{M-1}(d_{M-1}|\nu_1(\cdot|\alpha,\delta,\pi_1)) = M,$

$V_1^{M-1}(r_{M-1}|\nu_1(\cdot|\alpha,\delta,\pi_1)) = \delta(1-\alpha)(M+2) + \delta\alpha(M-1) + (1-\delta)(M-1) = M-1+3\delta(1-\alpha).$

Thus r_{M-1} is a best response provided $\delta(1-\alpha) \geqslant \frac{1}{3}$. To complete the proof we need to show that $s_i^*(\rho) = r_\rho$ is the preferred action for all $\rho \in H_i$ with $i = 1, 2$.

Inductive step Consider node ρ. Assume ρ is on the equilibrium path. We make the inductive hypothesis that r_κ is a best response at all nodes κ, $\rho < \kappa < M-1$. There are two cases to consider:

Case 1 $\rho = 2\tau + 1$ Player 1 has the move. The expected value of her pay-offs are:

$V_1^\rho(d_\rho|\nu_1(\cdot|\alpha,\delta,\pi_1)) = \rho + 1, \qquad V_1^\rho(r_\rho|\nu_1(\cdot|\alpha,\delta,\pi_1)) = \delta(1-\alpha)(M+2) + \delta\alpha\rho + (1-\delta)(M-1).$

Thus r_ρ is a best response provided $(1-\delta\alpha)M + 2\delta(1-\alpha) - (1-\delta) \geqslant (1-\delta\alpha)\rho + 1,$

$\Leftrightarrow (1-\delta\alpha)(M-\rho) \geqslant 2 - 3\delta + 2\delta\alpha$. Now $(1-\delta\alpha)(M-\rho) \geqslant 3(1-\delta\alpha) = 3 - 3\delta\alpha$. Thus a sufficient condition is, $3 - 3\delta\alpha \geqslant 2 - 3\delta + 2\delta\alpha \Leftrightarrow 1 \geqslant 2\delta\alpha - 3\delta(1-\alpha)$, which always holds since $\delta(1-\alpha) \geqslant \frac{1}{3}$.

Case 2 $\rho = 2\tau, 1 \leqslant \tau \leqslant M-2$. Player 2 has the move. The expected value of her pay-offs are:

$V_2^\rho(d_\rho|\nu_1(\cdot|\alpha,\delta,\pi_1)) = \rho + 1, \qquad V_2^\rho(r_\rho|\nu_1(\cdot|\alpha,\delta,\pi_1)) = \delta(1-\alpha)(M+1) + \delta\alpha\rho + (1-\delta)(M+1).$

[26] This will be proved once we have completed the induction.

Thus r_ρ is a best response provided: $(1-\delta\alpha)(M+1)+\delta\alpha\rho \geqslant \rho+1 \Leftrightarrow (1-\delta\alpha)(M-\rho) \geqslant \delta\alpha$. Since $M-\rho \geqslant 2$, a sufficient condition for r_ρ to be a best response is $2-2\delta\alpha \geqslant \delta\alpha \Leftrightarrow \delta\alpha \leqslant \frac{2}{3}$. Now $\delta(1-\alpha) \geqslant \frac{1}{3} \Rightarrow (1-\delta)+\delta\alpha \leqslant \frac{2}{3}$. Thus r_ρ is a best response under the given assumptions.

Having considered all possible cases we have established the inductive step. Thus there exists an equilibrium in which cooperation continues until the final node when $\delta(1-\alpha) \geqslant \frac{1}{3}$. Moreover if $\delta(1-\alpha) > \frac{1}{3}$ then r_ρ is the only best response at the relevant nodes, which establishes uniqueness of the equilibrium. ■

Proof of Proposition 5.2 We shall proceed by (backward) induction. At node the final node M, d_M is a dominant strategy. Now consider the decision at node $M-1$. Assume this node is off the equilibrium path. (This assumption will be confirmed when the proof is complete.)

Recall that, at nodes off the equilibrium path, the GBU updated preferences may be represented by the function, $W(a)=(1-\alpha)\max u(a)+\alpha\min u(a)$. Player 1 moves at this node. Her (Choquet) expected pay-offs from continuing are:

$V_1^{M-1}(r_{M-1}|\nu_1(\cdot|\alpha,\delta,\pi_1)) = (1-\alpha)(M+2)+\alpha(M-1) = M+2-3\alpha < M$
$= V_1^{M-1}(d_{M-1}|\nu_1(\cdot|\alpha,\delta,\pi_1))$, since $\alpha \geqslant \frac{2}{3}$ we may conclude that d_{M-1} is a best response.

Inductive step The inductive hypothesis is that d_κ is a best response at all nodes $M-1>\kappa>\rho>1$. Now consider the decision at node ρ. Assume this node is off the equilibrium path and that player i has the move at node ρ. His/her expected pay-offs are given by:

$$V_i^\rho(d_\rho|\nu_1(\cdot|\alpha,\delta,\pi_i)) = \rho+1, \qquad V_i^\rho(r_\rho|\nu_1(\cdot|\alpha,\delta,\pi_i)) = (1-\alpha)(\rho+3)+\alpha\rho = \rho+3(1-\alpha).$$

(Player i perceives no ambiguity about his/her own move at node $\rho+2$.) Thus d_ρ is a best response provided: $1 \geqslant 3(1-\alpha) \Leftrightarrow \alpha \geqslant \frac{2}{3}$. This establishes by induction that d_ρ is a best response at all nodes ρ such that $1<\rho\leqslant M$, provided they are off the equilibrium path.

Node 1 Finally we need to consider the initial node, which is different since it is on the equilibrium path. Player 1 has to move at this node. Her expected pay-offs are:

$V_1^1(d_1|\nu_1(\cdot|\alpha,\delta,\pi_1)) = 2, \qquad V_1^1(r_1|\nu_1(\cdot|\alpha,\delta,\pi_1)) = \delta(1-\alpha)4+\delta\alpha+(1-\delta) = 3\delta(1-\alpha)+1.$

Since $\alpha \geqslant \frac{2}{3}$ implies $\delta(1-\alpha) \leqslant \frac{1}{3}$, which implies that d_1 is a best response at the initial node. This confirms our hypothesis that subsequent nodes are off the equilibrium path. The result follows. ■

Proposition A.1 *For $\alpha < \frac{2}{3}$ and $\delta(1-\alpha) < \frac{1}{3}, \Gamma$ does not have a singleton CP-EUA.*

The proof of Proposition A.1 follows from Lemmas A.3, A.4 and A.5.

Lemma A.3 *Assume $\alpha < \frac{2}{3}$ then at any node $\tau, M > \tau \geqslant 2$, which is off the equilibrium path, the player to move at node τ will choose to play right, i.e. r_τ.*

Proof. We shall proceed by (backward) induction. To start the induction consider the decision at node $M-1$.

Node $M-1$ Assume this node is off the equilibrium path. Player 1 has the move. Her expected payoff from continuing is, $V^1\left(r_{2M-1}|\nu_1\left(\cdot|\alpha,\delta,\pi_1\right)\right) = (1-\alpha)(M+2) + \alpha(M-1) = M+2-3\alpha \geqslant M = V^1\left(d_{M-1}|\nu_1\left(\cdot|\alpha,\delta,\pi_1\right)\right)$, since $\alpha < \frac{2}{3}$.

Inductive step Since τ is off the equilibrium path so are all nodes which succeed it. The inductive hypothesis is that r_κ is a best response at all nodes $M-1 > \kappa > \rho \geqslant \tau$. Now consider the decision at node ρ. First assume that Player 2 has the move at node ρ, which implies that ρ is an even number. His expected pay-offs are given by:

$V^2\left(d_\rho|\nu_2\left(\cdot|\alpha,\delta,\pi_2\right)\right) = \rho+1, \qquad V_2^\rho\left(r_\rho|\nu_2\left(\cdot|\alpha,\delta,\pi_2\right)\right) = (1-\alpha)(M+1)+\alpha\rho.$

Thus r_ρ is a best response provided: $\quad (1-\alpha)(M+1-\rho) \geqslant 1$. Note that $\rho \leqslant M-2$ and since $\alpha < \frac{2}{3}$, $3(1-\alpha) > 1$ hence $(1-\alpha)(M+1-\rho) \geqslant 1$ which establishes that right is a best response in this case.

Now assume that Player 1 has the move at node ρ. Her expected pay-offs are given by:

$V_1^\rho\left(d_\rho|\nu_1\left(\cdot|\alpha,\delta,\pi_1\right)\right) = \rho+1, \qquad V_1^\rho\left(r_\rho|\nu_1\left(\cdot|\alpha,\delta,\pi_1\right)\right) = (1-\alpha)(M+2)+\alpha\rho.$

The analysis for Player 2 shows that r_ρ is also a best response in this case.

This establishes the inductive step. The result follows. ■

Lemma A.4 *Assume $\alpha < \frac{2}{3}$ and $\delta(1-\alpha) < \frac{1}{3}$. Let Γ be a M-stage centipede game, where $M \geqslant 4$. Then there does not exist a singleton equilibrium in which Player 1 plays d_1 at node 1.*

Proof. Suppose if possible that there exists a singleton equilibrium in which Player 1 plays d_1 at node 1. Then all subsequent nodes are off the equilibrium path. By Lemma A.3 players will choose right, r_ρ, at such nodes. Given this Player 1's expected pay-offs at node 1 are:

$V_1^1\left(d_1|\nu_1\left(\cdot|\alpha,\delta,\pi_1\right)\right) = 2,$

$V_1^1\left(r_1|\nu_1\left(\cdot|\alpha,\delta,\pi_1\right)\right) = \delta(1-\alpha)(M+2) + \delta(1-\alpha)\cdot 1 + (1-\delta)(M-1).$

Thus d_1 is a best response if $2 \geqslant (1-\delta\alpha) M + 3\delta(1-\alpha) + \delta - 1$

$\Leftrightarrow 3 - 3\delta(1-\alpha) - \delta \geqslant (1-\delta\alpha) M \geqslant 4 - 4\delta\alpha, \qquad$ since $M \geqslant 4$.

$\Leftrightarrow 7\delta\alpha - 4\delta \geqslant 1$. Since $\alpha < \frac{2}{3}, 1 > \frac{2}{3}\delta = \frac{14}{3}\delta - 4\delta > 7\delta\alpha - 4\delta$, hence d_1 cannot be a best response at node 1. ■

Lemma A.5 *Assume $\alpha < \frac{2}{3}$ and $\delta(1-\alpha) < \frac{1}{3}$. Let Γ be a M-stage centipede game, where $M \geqslant 4$. Then there does not exist a singleton equilibrium in which Player 1 plays r_1 at node 1.*

Proof. Suppose if possible such an equilibrium exists. Let τ denote the first node at which a player fails to cooperate. Since cooperation will definitely break down at or before node M, we know that $1 < \tau \leqslant M$. There are three possible cases to consider.

Case 1 $\quad \tau = M$. Consider the decision of Player 1 at node $M-1$. Her expected pay-offs are:

$V_1^{M-1}(d_{M-1}|\nu_1(\cdot|\alpha,\delta,\pi_1)) = M,$

$V_1^{M-1}(r_{M-1}|\nu_1(\cdot|\alpha,\delta,\pi_1)) = \delta(1-\alpha)(M+2) + \delta\alpha(M-1) + (1-\delta)(M-1)$

$= M + 3\delta(1-\alpha) - 1.$

However $\delta(1-\alpha) < \frac{1}{3}$, implies that r_{M-1} is not a best response. Thus it is not possible that $\tau = M-1$.

Case 2 $\quad \tau \leqslant M-1$ **and** $\tau = 2k+1$ $\quad$ This implies that Player 1 moves at node τ and that all subsequent nodes are off the equilibrium path. By Lemma A.3, right is a best response at these nodes. Hence her expected pay-offs are:

$V_1^{\tau}(d_\tau|\nu_1(\cdot|\alpha,\delta,\pi_1)) = \tau + 1, \qquad V_1^{\tau}(r_\tau|\nu_1(\cdot|\alpha,\delta,\pi_1)) = \delta(1-\alpha)(M+2) + \delta\alpha\tau + (1-\delta)(M-1).$

Thus d_τ is a best response provided,

$\tau + 1 \geqslant (1-\delta\alpha) M + 2\delta(1-\alpha) + \delta\alpha\tau - (1-\delta) \Leftrightarrow 2 + 2\delta\alpha - 3\delta \geqslant (1-\delta\alpha)(M-\tau).$

Since $M - \tau \geqslant 3$, a necessary condition for d_τ to be a best response is:

$5\delta\alpha - 3\delta \geqslant 1 \Leftrightarrow 2\delta\alpha \geqslant 1 + 3\delta(1-\alpha)$. However the latter inequality cannot be satisfied since $\alpha < \frac{2}{3}, 3\delta(1-\alpha) > \delta \geqslant \delta\alpha$, and $\delta\alpha < 1$. Hence it is not possible that τ is odd.

Case 3 $\quad \tau \leqslant M-2$ **and** $\tau = 2k$ $\quad$ This implies that Player 2 moves at node τ and that all subsequent nodes are off the equilibrium path. His pay-offs are

$V_2^{\tau}(d_\tau|\nu_2(\cdot|\alpha,\delta,\pi_2)) = \tau + 1, \qquad V_2^{\tau}(r_\tau|\nu_2(\cdot|\alpha,\delta,\pi_2)) = \delta(1-\alpha)(M+1) + \delta\alpha\tau + (1-\delta)(M+1).$

For d_τ to be a best response we need

$\tau + 1 \geqslant (1-\delta\alpha) M + 1 - \delta\alpha + \delta\alpha\tau \Leftrightarrow \delta\alpha \geqslant (1-\delta\alpha)(M-\tau).$

Since $\alpha < \frac{2}{3}$, $\delta\alpha < \frac{2}{3}$ and $(1 - \delta\alpha) > \frac{1}{3}$. In addition $M - \tau \geqslant 2$, which implies that this inequality can never be satisfied. Thus τ cannot be even.

Having considered all possibilities we may conclude that there is no pure equilibrium where Player 1 plays r_1 at node 1. ■

Proposition 5.3 *If $\alpha < \frac{2}{3}$ and $\delta(1-\alpha) < \frac{1}{3}$, there is a CP-EUA in which the support of Player 1's (resp. 2's) beliefs is $\left\{s_2^M, s_2^{M-2}\right\}$ (resp. $\left\{s_1^{M-1}, s_1^{M+1}\right\}$). The game will end at at $M-2$ or M with Player 2 exiting or at $M-1$ with Player 1 exiting.*

Proof. Assume that Player 1's beliefs are a neo-additive capacity based on the additive probability π_1 defined by $\pi_1(s_2^M) = p$, for $\pi_1(s_2^{M-2}) = 1-p$, $\pi_1(s_2) = 0$ otherwise, where $p = \frac{\delta(2-3\alpha)}{(1-\delta)}$. Likewise assume that Player 2's beliefs are a neo-additive capacity based on the additive probability π_2 defined by $\pi_2\left(s_1^{M+1}\right) = q$, for $\pi_2\left(s_1^{M-1}\right) = 1-q$, $\pi_2(s_1) = 0$ otherwise, where $q = \frac{1-3\delta(1-\alpha)}{3(1-\delta)}$.

First consider the updated beliefs. At any node ρ, $0 \leqslant \rho \leqslant M-2$, the updated beliefs are a neo-additive capacity with the same δ and α. The new probability π' is the restriction of the prior probability π to the set of continuation strategies. At node $M-1$, the GBU rule implies the updated beliefs are a neo-additive capacity with the same α, the updated δ given by $\delta_1' := \frac{\delta}{\delta+(1-\delta)p}$. The updated π assigns probability 1 to s_2^M.[27] We do not need to specify the beliefs at node M. Player 2 has a dominant strategy at this node which he always will choose.

Player 2 Consider Player 2's decision at node $M-2$. His actions yield pay-offs, $V_2^{M-2}(d_{M-2}|\nu_2(\cdot|\alpha,\delta,\pi_2))$ and $V_2^{M-2}(r_{M-2}|\nu_2(\cdot|\alpha,\delta,\pi_2))$. For both to be best responses they must yield the same expected payoff, which occurs when $q = \frac{1-3\delta(1-\alpha)}{3(1-\delta)}$, by equation (4). Since $\delta(1-\alpha) \leqslant \frac{1}{3}$ by hypothesis, $q \geqslant 0$. In addition $q \leqslant 1 \Leftrightarrow 3(1-\delta) \geqslant 1 - 3\delta(1-\alpha)$, which holds since $\alpha < \frac{2}{3}$ implies $(1-\alpha) > \frac{1}{3}$ thus $1 - 3\delta(1-\alpha) < 3 - 3\delta$.

Player 1 Now consider Player 1's decision at node $M-1$. His updated δ is given by: $\delta_1' = \delta(p \mid m') := \frac{\delta}{\delta+(1-\delta)p}$. His strategies yield pay-offs $V_1^{M-1}(d_{M-1}|\nu_1(\cdot|\alpha,\delta,\pi_1))$ and $V_1^{M-1}(r_{M-1}|\nu_1(\cdot|\alpha,\delta,\pi_1))$. For both of these strategies to be best responses they must yield the same expected utility which occurs when $p = \frac{2\delta-3\delta\alpha}{1-\delta}$, by equation (5). Since $\frac{2}{3} > \alpha$ by hypothesis, $p \geqslant 0$. Moreover $1 \geqslant p \Leftrightarrow 1-\delta \geqslant 2\delta - 3\delta\alpha \Leftrightarrow \frac{1}{3} \geqslant \delta(1-\alpha)$, which also holds by hypothesis. ■

[27] The form of the updated beliefs follows from the formulae in Eichberger, Grant, and Kelsey (2010).

A.3 Bargaining

Proof of Proposition 6.1 To establish that the strategy profile specified in the statement of Proposition 6.1 constitutes a CP-EUA, we work backwards from the end of the game.

At $t = 3$. Since this is the last period, it is a best response for player 2 to accept *any* proposed division $\langle z, 1 - z \rangle$ by player 1. So in any consistent planning equilibrium, player 1 will propose a $\langle 1, 0 \rangle$ division after any history.

At $t = 2$. If player 1 rejects player 2's proposed division $\langle y, 1 - y \rangle$, then the game continues to period 3 where we have established player 1 gets the entire cake. However, the neo-expected payoff to player 1 in period 2 of *rejecting* 2's offer is $(1 - \alpha)\beta$. This is because by rejecting player 2's proposed division, player 1 is now in an "off-equilibrium" history h, with $\delta(h) = 1$. Thus her neo-expected payoff is simply an $(\alpha, 1 - \alpha)$-weighting of the best and worst outcome that can occur (given his own continuation strategy). Hence for player 2's proposed division $\langle y, 1 - y \rangle$ to be accepted by player 1 requires $y \geqslant (1 - \alpha)\beta$.

So the neo-expected payoff for player 2 of proposing the division $\langle (1 - \alpha)\beta, 1 - (1 - \alpha)\beta \rangle$ in round 2 is $(1 - \delta_2)(1 - (1 - \alpha)\beta) + \delta_2 (1 - \alpha) \max\{1 - (1 - \alpha)\beta, \beta\}$, where $\delta_2 = \delta / (\delta + (1 - \delta)\pi)$ if $\langle x, 1 - x \rangle = \langle 1, 0 \rangle$, =1 otherwise. Notice that the 'best' outcome for player 2 is either $(1 - (1 - \alpha)\beta)$ *or* possibly β which is what she would secure if player 1 rejected his proposed division and then followed that by proposing in period 3 the (extraordinarily generous) division $\langle 0, 1 \rangle$.

If player 2 proposes $\langle y, 1 - y \rangle$ with $y < (1 - \alpha)\beta$, then according to our putative equilibrium, player 1 rejects and player 2 receives a payoff of zero. However, his neo-expected payoff of that is $(1 - \alpha)\max\{1 - y, \beta\}$, where $1 - y$ is what he would get if player 1 actually accepted his proposed division. Thus his best deviation is the proposal $\langle y, 1 - y \rangle = \langle 0, 1 \rangle$.

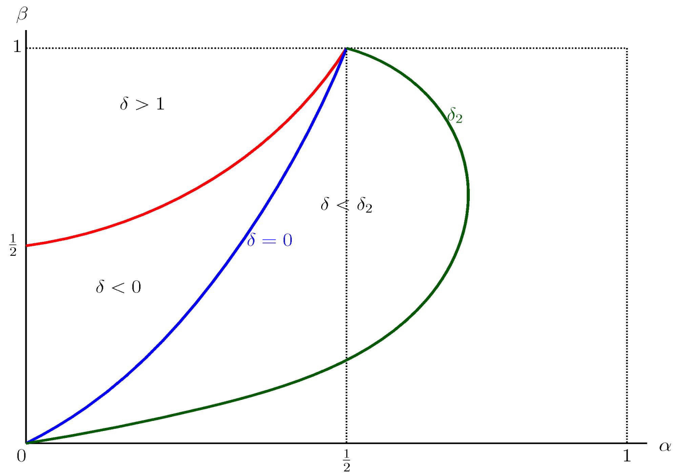
So we require:

$$(1 - \delta_2)(1 - (1 - \alpha)\beta) + \delta_2 (1 - \alpha) \max\{1 - (1 - \alpha)\beta, \beta\} \geqslant (1 - \alpha). \tag{10}$$

This is equivalent to

$$\bar{\delta}(\alpha, \beta) := \frac{\alpha - (1 - \alpha)\beta}{1 - (1 - \alpha)[\max\{1 - (1 - \alpha)\beta, \beta\} + \beta]} \delta_2.$$

The following figure shows the region of parameters (α, β) satisfying this constraint.



parameter region: $\delta_2 \leq \overline{\delta}(\alpha, \beta)$

Clearly, when $\delta_2 = 1$ (that is, when we are already off the equilibrium path), (10) cannot hold for any α strictly less than 1. So in that case, player 2's best action is indeed to propose $\langle y, 1-y \rangle = \langle 0, 1 \rangle$, which is what our putative equilibrium strategy specifies in these circumstances.

At $t = 1$, if player 2 rejects player 1's proposed division $\langle x, 1-x \rangle$, then the game continues to $t = 2$ in which case he offers player 1 the division $\langle y^*, 1-y^* \rangle = \langle (1-\alpha)\beta, 1-(1-\alpha)\beta \rangle$, which according to her strategy she accepts.

On the equilibrium path, $\delta_2 = \delta < 1$, the neo-expected payoff for player 2 in period 1 of rejecting player 1's proposed division $\langle x, 1-x \rangle$ is $(1-\delta)(1-(1-\alpha)\beta)\beta + \delta\alpha$. In this putative equilibrium Player 1 proposes the division $\langle x^*, 1-x^* \rangle = \langle 1, 0 \rangle$ which player 2 rejects, since she anticipates an offer of $\langle (1-\alpha)\beta, 1-(1-\alpha)\beta \rangle$ in period 2. So her period 1 neo-expected payoff is $(1-\delta)(1-\alpha)\beta^2 + \delta(1-\alpha)$, since the 'best' thing that could happen is player 2 actually accepts her proposed division $\langle 1, 0 \rangle$ today!

Alternatively, player 1 might propose a division $\langle \tilde{x}, 1-\tilde{x} \rangle$, that would just entice player 2 to accept today. That is, she could propose $\langle \tilde{x}, 1-\tilde{x} \rangle$ where

$$\tilde{x} = 1 - \left[(1-\delta)(1-(1-\alpha)\beta) + \delta(1-\alpha)\beta \max\{1-(1-\alpha)\beta, \beta\} \right].$$

The neo-expected payoff for player 1 of this deviation is: $\delta(1-\alpha)\max\{\tilde{x}, \beta\}$. In order for it not to be "profitable", we require $(1-\delta)(1-\alpha)\beta^2 + \delta(1-\alpha) \geqslant \delta(1-\alpha)\max\{\tilde{x}, \beta\}$, which holds since $\max\{\tilde{x}, \beta\} < 1$. ■