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Objective and subjective rationality and decisions with the best and worst case in mind

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Abstract
We study decision under uncertainty in an Anscombe–Aumann framework. Two binary relations characterize a decision-maker: one (in general) incomplete relation, reflecting her objective rationality, and a second complete relation, reflecting her subjective rationality. We require the latter to be an extension of the former. Our key axiom is a dominance condition. Our main theorem provides a representation of the two relations. The objectively rational relation has a Bewley-style multiple prior representation. Using this set of priors, we fully characterize the subjectively rational relation in terms of the most optimistic and most pessimistic expected utilities.

Keywords Ambiguity · Incomplete preferences · Optimism · Pessimism

1 Introduction

It is well known that ambiguity affects decision-making, often in important ways. It is less well understood, however, where ambiguity preferences come from; there are many theories of ambiguity preference and modeling approaches are diverse. Although it is common to model agents as pessimistic or cautious in the face of ambiguity, this is far from the only attitude with theoretical plausibility or empirical

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support. Jürgen Eichberger—in whose honor this special issue appears—was an early proponent of the view that some agents are instead optimistic. He developed models accommodating a wider range of ambiguity attitudes, and showed that this kind of heterogeneity can have significant economic implications (Chateauneuf et al. 2007; Eichberger et al. 2008).

We are in full agreement with Eichberger on this point; in a companion paper (Grant et al. 2020), we develop a meta-utility theory in which the worst-case and best-case expected utilities of an act (with respect to the set of priors C and the utility index u) serve as sufficient statistics to describe a decision-maker's preferences. The worst-case and best-case expected utilities, being cardinal, act like any other quantities, and can serve as inputs to an ordinal utility function. For example, the arithmetic weighted average expected utility model of Hurwicz (1951) is a meta-utility function in which the worst-case and best-case expected utilities are perfect substitutes. The minimum expected utility theory of Wald (1950), Gilboa and Schmeidler (1989) is an important special case, and Binmore's (2009) geometric weighted average expected utility theory is likewise subsumed. We dub this family of preferences *Ordinal Hurzwicz Expected Utility*, as they extend Hurwicz to a more general class of two-input meta-utility functions. An important insight from this theory is that the marginal rate of substitution between the worst-case and best-case expected utilities can serve as a general definition of ambiguity attitude, allowing us to compare ambiguity attitude across (even more complex) theories and across agents. Our purpose in this article is to provide an alternative foundation for preferences that admit the type of representation that we, like Eichberger, have in mind.

Regarding the source of such preferences, our theory fits naturally with the approach of Gilboa et al. (2010) (hereafter GMMS), who work in the tradition of Seidenfeld et al. (1995). GMMS present a novel axiomatization of the min expected utility (minEU) or multiple priors model of Gilboa and Schmeidler (1989). They suppose that there are two primitive relations, $(\succsim^*, \hat{\succsim})$. The first, $\succsim^*$, is an incomplete order, thought of as capturing the decision-maker's (DM's) objectively rational choices. That is, $\succsim^*$ determines which options are provably better than which others (with respect to the agent's fixed beliefs and risk preferences). The second, $\hat{\succsim}$, is an extension of $\succsim^*$, thought of as capturing subjectively rational choices, that is, preferences which are not provably mistaken. Because $\hat{\succsim}$ extends $\succsim^*$, the two relations are consistent in the sense that any objectively rational preference is also subjectively rational. The crucial axiom of GMMS that enables them to obtain a minEU representation is that the subjectively rational relation is cautious, in the sense that decision-makers default to certainty: any ambiguous act that is not objectively preferable to a risky act is subjectively worse. They interpret their result as providing a possible account of how minEU behavior might emerge, namely from the cautious completion of incomplete (but objectively rational) preferences. They suggest (in Sect. 3) that weakening this axiom could accommodate other representations of interest, a challenge that we accept.[1]

In this article, we work within an Anscombe–Aumann framework and follow the GMMS approach to provide an alternative foundation for our more permissive meta-utility theory. We show that the completion of the objectively rational preference relation by means of a natural dominance argument—in which the dominance that is applied is defined with respect to acts that exhibit no ambiguity—results in a subjectively rational preference relation that has an Ordinal Hurzwicz Expected Utility representation. Hence (echoing GMMS), we contend that behavior conforming to the maximization of Ordinal Hurwicz Expected Utility can emerge from such a dominance completion of objectively rational preferences.

Just as there are non-Hurwicz approaches to ambiguity, there are other ways of generalizing the GMMS model to reflect them, and some have been investigated. After presenting our approach, we discuss other ways of extending incomplete preferences in an Anscombe–Aumann framework to characterize existing models of decision under ambiguity. Specifically, we compare our model to those of Cerreia-Vioglio (2016) and Cerreia-Vioglio et al. (2020). While the primary interpretation of the latter is somewhat different, considering the relationships between our axioms and these other extensions can highlight the common foundations and key differences among theories that are typically studied in isolation. We view the comparisons of these axioms as helpful in providing a generalized understanding of models of decision under ambiguity.

The theories which we discuss take a static viewpoint. A future direction could address a DM's updating of beliefs as new information arises. As is well known, Bayesian updating in a multiple priors world can lead to surprising behavior, such as the set of posteriors dilating to a superset of the set of priors (see Seidenfeld and Wasserman 1993). Some results in this direction are in Manski (1981), who considers Bayesian updating given subjective domains of probability measures. Manski, like us, focuses on the worst- and best-case expected utilities (see p. 63 in particular). For a more recent approach closely tied to GMMS, we refer the reader to Faro and Lefort (2019).

2 Framework

We begin with Fishburn's (1970) rendition of the Anscombe and Aumann (1963) framework. Let X denote the set of final outcomes. A (simple) lottery P is a probability density function from X to [0, 1] with finite range, satisfying $\sum_{x \in X} P(x) = 1$. Let L denote the set of lotteries. As is standard, we endow L with the mixture operation in which, for any pair of lotteries $P, Q \in L$ and any $\lambda \in [0, 1]$, $\lambda P + (1 - \lambda)Q$ corresponds to the lottery $R \in L$ by setting $R(x) := \lambda P(x) + (1 - \lambda)Q(x)$.

[1] A similar axiomatization to GMMS is in Ghirardato et al. (2004). For other approaches based on extending incomplete preferences, see Aumann (1962), Bridges and Mehta (1995), Bewley (2002), Ok (2002), Dubra et al. (2004).

As GMMS do, we take the set of *states of the world* S to be endowed with an algebra Σ of *events*. The set $\Delta(\Sigma)$ of (finitely additive) *probabilities* on Σ (with generic element π) is endowed with the eventwise convergence topology. Therefore, we take F to be the set of Anscombe–Aumann acts; in particular, each $f \in F$ is a Σ-measurable function from S to L with finite range. Therefore, for any lottery P, $f^{-1}(P) := \{s \in S : f(s) = P\}$. We endow F with the mixture operation in which, for any $f, g \in F$ and any $\lambda \in [0,1]$, $\lambda f + (1-\lambda)g$ is the act $h \in F$ given by $h(s) = \lambda f(s) + (1-\lambda)g(s)$.

We shall identify the set of constant acts with the set of lotteries L. That is, any lottery $P \in L$ will also denote, with slight abuse of notation, the constant act $h \in F$ in which $h(s) = P$ for all $s \in S$. Notice that constant acts by definition are unaffected by the uncertainty as to which state s in S obtains.

For a function $u : X \to \mathbb{R}$ and a lottery (or constant act) $P \in L$, we follow the notation of GMMS and write $E_P u = \sum_{x \in X} P(x)u(x)$.

3 GMMS axioms and the minEU rule

Following GMMS, we consider a DM characterized by two binary relations, $\succsim^*$ and $\hat{\succsim}$, the former reflecting choices that the DM can convince others she is right to make, and the latter reflecting choices that are subjectively rational.

We begin with a listing of the axioms considered by GMMS.

Preorder:	$\succsim$ is reflexive and transitive.
Monotonicity:	For every $f, g \in F$, $f(s) \succsim g(s)$ for all $s \in S$ implies $f \succsim g$.
Continuity:	For all $e, f, g, h \in F$, the set $\{\lambda \in [0,1] : \lambda f + (1-\lambda)g \succsim \lambda h + (1-\lambda)e\}$ is closed in [0, 1].
Nontriviality:	There exist $f, g \in F$, such that $f \succ g$.
Completeness:	For every $f, g \in F$, $f \succsim g$ or $g \succsim f$.
C-Completeness:	For every $P, Q \in L$, $P \succsim Q$ or $Q \succsim P$.
Independence:	For every $f, g, h \in F$ and every $\alpha \in [0,1]$, $f \succsim g$ implies $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$.
Consistency:	$f \succsim^* g$ implies $f \ \hat{\succsim} \ g$.
Default to Certainty:	For $f \in F$ and $P \in L$, $f \not\succsim^* P$ implies $P \mathrel{\hat{\succ}} f$.

The first four conditions (which GMMS call their basic conditions) might be seen as naturally applying to both preference relations. However, GMMS show in their Theorem 4 (which we restate as Theorem 1 below) that if the DM satisfies Default to Certainty (rather than Caution, its weak preference analogue), it is necessary to require monotonicity only of the objective preference $\succsim^*$. This also holds for Theorem 2, our main representation result, as we show below.

With respect to Completeness, GMMS argue that only the subjectively rational relation $\hat{\succsim}$ needs to satisfy it. The objectively rational relation, on the other hand, needs to be complete only on its restriction to the set of constant acts. That is, $\succsim^*$

needs to satisfy only C-Completeness rather than the considerably more demanding Completeness. However, it is required to satisfy independence.[2]

The final two building blocks in their axiomatization of minEU preferences are the consistency requirement that the subjectively rational relation agrees with the objectively rational one, and that the completion of the subjectively rational relation may be thought of as having been achieved through a default to certainty procedure of strictly preferring a constant act over a general act whenever the objectively rational relation has not already determined that preference.

Theorem 1 (GMMS Theorem 4:) *The following statements are equivalent:*

(i) $\succsim^*$ *satisfies Preorder, C-Completeness, Continuity, Nontriviality, Monotonicity, and Independence;* $\hat{\succsim}$ *satisfies Preorder, Completeness, and Continuity; and jointly, they satisfy Consistency and Default to Certainty.*

(ii) *There exist a nonempty closed and convex set* C *of probabilities on* Σ *and a nonconstant function* $u : X \to \mathbb{R}$, *such that, for every* $f, g \in F$:

(a) $f \succsim^* g$ *if and only if*

$$\forall\, \pi \in C\,, \sum_{P \in L} \pi(f^{-1}(P)) E_P u \geqslant \sum_{P \in L} \pi(g^{-1}(P)) E_P u\,;$$

(b) $f\ \hat{\succsim}\ g$ *if and only if*

$$\min_{\pi \in C} \sum_{P \in L} \pi(f^{-1}(P)) E_P u \geqslant \min_{\pi \in C} \sum_{P \in L} \pi(g^{-1}(P)) E_P u\,.$$

4 A meta-utility of the minEU and the maxEU

In this section, we consider a DM whose subjective preference relation may exhibit a less extreme and potentially more-nuanced attitude toward the ambiguity she perceives there to be. To keep the model tractable, however, we characterize her evaluation of the induced state-contingent utility of an act in terms of a meta-utility defined over the utility of the best constant act it dominates (that is, its minEU according to the associated representation of the objective relation) and the utility of the worst constant act that dominates it (that is, its maxEU according to the associated representation of the objective relation).

[2] Strictly speaking, the version of continuity which we have listed is stronger than the one that appears in GMMS, while the independence axiom is weaker. However, as GMMS note in their final technical remarks on p 769, Shapley and Baucells (1998) have shown that, in the presence of Preorder, this stronger continuity axiom and this weaker independence axiom imply the independence axiom that GMMS use. Conversely, the representation of $\succsim^*$ that they obtain in their theorems implies the stronger continuity axiom.

An illustration may help to clarify the types of DMs our characterization describes. It will not be possible to convince every DM that they should limit attention to the worst case, as implied by the Default to Certainty axiom of GMMS (and its weak version, Caution). An historical example shows that even highly rational individuals may hold persistently different attitudes toward ambiguity, even in momentous cases. We now know that detonating an atomic bomb does not immediately destroy the (entire) earth, but there was a time at which scientists could not rule out a positive probability of this occurrence, and had to decide whether to proceed with a test of the new technology as part of the Manhattan Project. As Buck (1959) and Davis (1968) relate, Edward Teller made calculations suggesting that detonating the bomb might cause the oceans to explode and vaporize the earth. Everyone involved had a strong preference to avoid this worst-case outcome, but also strongly desired the best-case outcome (the expected payoff if the Allies could develop the bomb before the Axis). The ultimate DM, Arthur Compton, determined the maximum acceptable chance of extinguishing life on earth to be 3 in 1 million. Although this probability could not be estimated precisely, the Manhattan Project scientists were able to bound the worst-case probability below Compton's threshold, but still viewed it as above 0. As we know, Compton ultimately found the ambiguous act of testing the bomb to be preferable. Not everyone agreed, however, despite plausibly having the same objective preferences (vaporizing earth $\prec$ losing the war $\prec$ developing the bomb before the Axis could). The ultimate decision was rather a result of the decision-making hierarchy. Capturing this form of reasoning in the current framework requires the ability to extend incomplete preferences in different directions, in ways that can allow for decisions to be made solely based on the best- and worst-case outcomes.

We achieve this by proposing the following dominance property. Suppose one act dominates another act in terms of the respective set of constant acts that each is ranked no worse than. Furthermore, suppose the former act also dominates the latter act in terms of the respective set of constant acts that each is ranked no better than. Then, we propose that such two-sided dominance should be inherited by the subjective relation.

Since this dominance is defined relative to the objective preferences over constant acts, we dub this dominance property C-Dominance.

C-Dominance: For any two acts f and g in F,
if, for all P in L, $g \succsim^* P \Rightarrow f \succsim^* P$ and $P \succsim^* f \Rightarrow P \succsim^* g$, then $f \ \hat{\succsim}\ g$.

Note that C-Dominance requires only that the DM subjectively prefer f to g; she need not objectively prefer f. This allows for possibilities such as the DM comparing f and g differently according to which setting she has in mind, despite never viewing g as strictly better than f (as in the incomplete preferences setting of Stecher (2008)).

To aid the exposition, we introduce the following notation. Fix a nonempty closed and convex set C of probabilities on Σ and a nonconstant function $u : X \to \mathbb{R}$. For each act $f \in F$, set:

$$\underline{u}^f_C := \min_{\pi \in C} \sum_{P \in L} \pi\big(f^{-1}(P)\big) E_P u\,,$$
$$\overline{u}^f_C := \max_{\pi \in C} \sum_{P \in L} \pi\big(f^{-1}(P)\big) E_P u\,.$$

We define what it means for preferences to admit our notion of a meta-utility representation as follows:

Definition 1 **(OHEU representation)** The pair of preference relations $(\succsim^*, \hat{\succsim})$ admits an *Ordinal Hurwicz Expected Utility* (OHEU) representation, if there exist a nonempty closed and convex set C of probabilities on Σ, a nonconstant function $u : X \to \mathbb{R}$, and a monotonic function $V : u(X) \times u(X) \to \mathbb{R}$ in which $V(w, w) = w$ for all $w \in u(X)$, such that for every $f, g \in F$:

(a) $f \succsim^* g$ if and only if

$$\forall\, \pi \in C\,, \sum_{P \in L} \pi(f^{-1}(P)) E_P u \geqslant \sum_{P \in L} \pi(g^{-1}(P)) E_P u\,;$$

(b) $f \hat{\succsim} g$ if and only if $V\Big(\underline{u}^f_C, \overline{u}^f_C\Big) \geqslant V\big(\underline{u}^g_C, \overline{u}^g_C\big)$.

In this context, by the monotonicity of V, we mean the following: if $x' \geq x$ and $y' \geq y$, then $V(x', y') \geq V(x, y)$, with strict inequality in V when there is strict inequality between *both* the xs and the ys.

To obtain such a convenient two-parameter meta-utility representation of the subjective preferences $\hat{\succsim}$, all we need to do is replace Default to Certainty with C-Dominance in Theorem 1.

Theorem 2 (OHEU) *The following statements are equivalent:*

(i) $\succsim^*$ *satisfies Preorder, C-Completeness, Continuity, Nontriviality, Monotonicity, and Independence;* $\hat{\succsim}$ *satisfies Preorder, Completeness, and Continuity; and jointly, they satisfy Consistency and C-Dominance.*

(ii) *The pair of preference relations* $(\succsim^*, \hat{\succsim})$ *admits an Ordinal Hurwicz Expected Utility representation.*

Proof Necessity of the axioms for an OHEU representation is straightforward and, therefore, omitted. Also, given the existence of such a V, we immediately obtain uniqueness from the requirement that V is monotone and that, for all w in the image of u, $V(w, w) = w$.

Similarly, sufficiency for part (a) in Definition 1 depends only on the properties of $\succsim^*$, and not on those of $\hat{\succsim}$ or the relationships between $\succsim^*$ and $\hat{\succsim}$. These properties are the same as assumed in Theorem 1, so part (a) immediately follows from this theorem. It, therefore, remains for us to show sufficiency for part (b).

The only difference in the hypotheses of Theorem 2 from those of Theorem 1 is the replacement of Default to Certainty with C-Dominance. Fix arbitrary $f, g \in F$. One side of C-Dominance requires, for each $P' \in L$:

$$P' \succsim^* f \Rightarrow P' \succsim^* g\,,$$

if and only if

$$\text{for all } \pi \in C, \sum_{P \in L} \pi\big(f^{-1}(P)\big) E_P u \leqslant E_{P'} u$$
$$\Rightarrow \text{for all } \pi \in C, \sum_{P \in L} \pi\big(g^{-1}(P)\big) E_P u \leqslant E_{P'} u\,.$$

Part (a) establishes that C is closed, convex, and nonempty, and our assumptions state that both preference relations are continuous and that $\hat{\succsim}$ extends $\succsim^*$. Therefore, the above condition implies:

$$E_{P'} u \geqslant \underline{u}_C^f \Rightarrow E_{P'} u \geqslant \underline{u}_C^g\,.$$

Therefore, in particular, for the constant act $P^* \in L$ for which $E_{P^*} u = \underline{u}_C^f$, it is then the case that $\underline{u}_C^f \geqslant \underline{u}_C^g$. As f has finite range and its codomain is L, there is always a best and worst lottery in the range of f. Such a P^*, therefore, always exists by the fact that L is closed under mixtures and from continuity.

Analogously, if for each $P' \in L$:

$$g \succsim^* P' \Rightarrow f \succsim^* P'\,,$$

then there exists $P^{**} \in L$ with $E_{P^{**}} u = \overline{u}_C^f$, and a parallel argument shows:

$$\overline{u}_C^g \leqslant \overline{u}_C^f.$$

This thus establishes that C-Dominance entails $f \,\hat{\succsim}\, g$ whenever $\underline{u}_C^f \geqslant \underline{u}_C^g$ and $\overline{u}_C^f \geqslant \overline{u}_C^g$, which means that in the representation, $V(\underline{u}_C^f, \overline{u}_C^f) \geqslant V(\underline{u}_C^g, \overline{u}_C^g)$. This, in turn, requires the subjective relation to view as indifferent any two acts f and g for which $\underline{u}_C^f = \underline{u}_C^g$ and $\overline{u}_C^f = \overline{u}_C^g$; hence, $V(\underline{u}_C^f, \overline{u}_C^f) = V(\underline{u}_C^g, \overline{u}_C^g)$. Or, in other words, for any act f, its associated pair of extreme expected utilities, $(\underline{u}_C^f, \overline{u}_C^f)$, constitutes a sufficient statistic that characterizes the indifference class in which this act resides. □

The form that the meta-utility function V takes is largely left open in the representation, since the purpose here is to accommodate heterogeneity in ambiguity preferences. Salient possibilities for V abound in the literature, however, as a companion paper elaborates (Grant et al. 2020). To see what V might look like, we consider how three specific meta-utility functions would evaluate an act f, for which we let $\underline{u}_C^f = 3$, and $\overline{u}_C^f = 9$.

According to the minEU theory (Wald 1950; Gilboa and Schmeidler 1989) discussed in the introduction, $V\left(\underline{u}_C^f, \overline{u}_C^f\right) = \underline{u}_C^f$. Hence, $V(3, 9) = 3$. Echoing Eichberger, we can instead imagine an agent who is much more optimistic, and evaluates the ambiguous act f as much better than a risky act with expected utility of 3. An α-minEU agent (Hurwicz 1951; Gul and Pesendorfer 2015) with (pessimism) $\alpha = \frac{1}{3}$, for example, may be represented by a meta-utility function $V\left(\underline{u}_C^f, \overline{u}_C^f\right) = \frac{1}{3}\underline{u}_C^f + \frac{2}{3}\overline{u}_C^f$. Then, $V(3,9) = 7$. Finally, Binmore characterizes a geometric-α-minEU agent, whose meta-utility function—again letting $\alpha = \frac{1}{3}$—would instead be $V\left(\underline{u}_C^f, \overline{u}_C^f\right) = (\underline{u}_C^f)^{\frac{1}{3}}(\overline{u}_C^f)^{\frac{2}{3}}$. Then, $V(3,9) \approx 6.24$.

Therefore, we see that we can represent agents who are optimistic or pessimistic not only to different degrees, but in different ways; the ambiguity attitude of the minEU and α-minEU agents are constant, in that their level of pessimism is fixed, whereas the geometric-α-minEU agent is more cautious than their α-minEU counterpart at this relatively low expected utility level, but becomes less and less so as $\underline{u}_C^f$ and $\overline{u}_C^f$ increase. Graphical illustrations and further discussion of how to characterize and compare ambiguity attitude in this setting can be found in Grant et al. (2020).

Also note that, for all of the above preferences, the meta-utility function must be such that $V(3,3) = 3$, $V(9,9) = 9$, and so forth; risky acts are valued at their expected utility. The functions may differ only in how they fill in the valuations for the acts "in between." Therefore, in fact, once C and u are fixed, the V representing each of these three preference types is unique. The uniqueness comes from the requirement that $V(w,w) = w$. Given an expected utility representation of risk preferences, imposing this requirement not only simplifies the representation, but also imposes a natural coherence between the expected utility and meta-utility components of the representation.

5 Discussion

Our OHEU representation rests on one key departure from GMMS: our weakening of Default to Certainty to C-Dominance. As this axiom does the heavy lifting for us, we conclude with some added interpretation of C-Dominance and its relationship to its counterparts in other approaches based on completing an incomplete preference relation.

One aspect of C-Dominance that is easy to overlook is that the premises are in terms of the objective preference relation $\succsim^*$, while its conclusion is in terms of its completion, the subjective relation $\hat{\succsim}$. To argue that act f dominates act g, the DM can call any $P \in L$ as a witness to testify in favor of f over g. If P objectively dominates f, then P immediately testifies that it also objectively dominates g. Conversely, if g objectively dominates P, then P immediately testifies that f also dominates P. Given that C-Completeness says that the objective preference is defined over all of L, the DM can provide what may appear to be comprehensive testimony that f is at least as good as g, so the reader might wonder why C-Dominance requires only that f is subjectively better than g.

We illustrate the reason with a three-color Ellsberg urn; this will be a running example throughout the discussion. Assume as usual that the probability of drawing a red ball is 1/3, that of drawing a black ball is between 0 and 2/3, and therefore, that of drawing a yellow ball is also between 0 and 2/3. Let f be an act that pays \$10 if a black ball is drawn and \$5 otherwise, and let g be an act that pays \$10 if a yellow ball is drawn and \$5 otherwise. For any $P \in L$ that pays \$10 or more with probability of at least 2/3 and \$5 or more otherwise, P objectively dominates both f and g, and any P' that pays at most \$5 is objectively dominated by both f and g. Therefore, $P \succsim^* f \Leftrightarrow P \succsim^* g$, and $g \succsim^* P' \Leftrightarrow f \succsim^* P'$.

Under some probabilities which the DM considers, f is strictly better than g, and under others, g is strictly better than f. The DM could not convince all others that f is at least as good as g or that g is at least as good as f. C-Dominance does not require this, but allows us to infer that, subjectively, $f \hat{\sim} g$.

An alternative weakening of the GMMS axioms is in Cerreia-Vioglio (2016), which provides a generalization of the uncertainty averse preferences discussed in Cerreia-Vioglio et al. (2011); these include variational preferences (Maccheroni et al. 2006), multiplier preferences (Hansen and Sargent 2001; Strzalecki 2011), and smooth preferences (Klibanoff et al. 2020). In addition to assuming a weaker version of independence (applying only to objective lotteries), Cerreia-Vioglio (2016) considers the following axiom, which he calls Weak Caution:

$$(\forall P \in L)\,(\exists P' \in L \text{ with } P' \hat{\succsim} P) f \not\succsim^* P \Rightarrow P' \hat{\succsim} f.$$

Note that C-Completeness implies that $P' \succsim^* P$. Weak Caution says that a DM can evaluate an act f by trying to compare it with a lottery $P \in L$. If f is not objectively better than P, then the DM can find some other $P' \in L$ which is better than P and which the DM would weakly subjectively prefer to f. In contrast to GMMS, a DM satisfying Weak Caution need not subjectively prefer a lottery P to an act f solely on the grounds that f is not objectively better than P. However, the inability to rank f above P means that there is some bound on what better lottery P' f could be as good as.

To understand the intuition of Weak Caution, consider again the standard three-color Ellsberg urn. Let f once more be an act that pays \$10 on black ball and \$5 otherwise. Let P be a lottery that pays \$5 on a red ball and \$5.01 otherwise. The DM cannot objectively prove that f is at least as good as P, because she cannot bound the probability of a black ball away from 0. Under Weak Caution, the DM does not necessarily subjectively default to viewing P as being at least as good as f. However, an improved lottery P' is associated with P, such that P' does weakly subjectively dominate f, along with any other act g that is not objectively better than P. For instance, the DM may subjectively prefer a lottery P' that pays \$5 on red and \$6 otherwise to any act such as f that is not objectively at least as good as P, even though f is not objectively worse than P'. This captures the idea of an uncertainty averse preference as defined in Cerreia-Vioglio et al. (2011), in the way that parallels how C-Dominance captures the idea of a generalized Hurwicz-style preference.

To summarize, Weak Caution extends GMMS' minEU model to incorporate several standard models of the uncertainty averse preference type, while C-Dominance extends GMMS to incorporate models that depend on the best- and worst-case expected utility, such as Hurwicz (1951) and Binmore (2009). As discussed in Chateauneuf et al. (2007) and Grant et al. (2020), Hurwicz-type models capture important special cases of the smooth and variational models, along with related models such as Siniscalchi (2009). By viewing the axiomatization of Cerreia-Vioglio (2016) alongside the current one, we can obtain a deeper understanding of the relationships among models in these two families.

Taken together, these results tell us how extending incomplete preferences in the Bewley (2002) tradition can characterize (in an Anscombe–Aumann world) different approaches to decision under ambiguity first developed in a Savage framework. Default to Certainty captures minEU; Weak Caution captures uncertainty averse preferences; C-Dominance captures Hurwicz-type preferences. It is natural to ask how far these extensions can go.

This task is taken up in Cerreia-Vioglio et al. (2020), who interpret the incomplete preference relation $\succsim^*$ as the DM's mental preferences, that is, what the DM actually prefers, and its completion as extending the DM's preferences to characterize observed behavior, whether an actual preference is present or not. Their axioms are very permissive; indifference in the behavioral relation (corresponding to our $\hat{\succsim}$) need not be transitive. This is in line with May (1954), who cites animal behavior literature demonstrating that sufficiently hungry rats prefer food to sex, sex to avoidance of pain, and avoidance of pain to food. May views mental preferences as arising from attributes, not inherently defined over the objects of choice. Their extension to the objects of choice arises from aggregation, which can lead to intransitivity for reasons familiar from multi-attribute decision-making and social choice theory.

Cerreia-Vioglio et al.'s (2020) crucial axiom capturing how the DM responds to a lack of real preference is Possibility, which is a much weaker counterpart of Default to Certainty than C-Dominance. Possibility says that any act g not weakly dominated by act f under $\succsim^*$ can potentially be chosen over f. This axiom, along with a restriction on consistency to hold only on the interior of upper contour sets, enables them to obtain an existential characterization, subsuming any choice behavior that could be consistent with EU on the restriction to the domain of $\succsim^*$.

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