Decomposable Splits, Coherent Beliefs, and the Ex Ante Optimality of Consistent Plans*

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Abstract

Consider a decision-maker who, like Gul and Pesendorfer's (2014) expected uncertain utility maximizer, has coherent beliefs represented by a probability (the decision-maker's 'prior') defined on (and only on) those events she deems unambiguous and thus measurable. We characterize the class of such decision-makers who deem measurable any event that along with its complement constitute what we dub a *decomposable split* of the state space, a property that is implied by but does not imply Savage's postulate **P2**. As a consequence the restriction of her preferences to acts that are measurable with respect to her prior, (that is, her *risk preferences*) need only exhibit betweenness. Thus, not only can our theory accommodate Ellsberg style choice patterns for decisions involving ambiguous prospects, it can also produce Allais style choice patterns for decisions over measurable (that is, "risky") prospects.

We extend our analysis to a dynamic setting in which a decision-maker with coherent beliefs is tasked with selecting an act from a fixed menu conditional on a signal. We assume her perception of the signal's realizations is suitably adapted to the events she can measure with her prior. Our main result is, given the decision maker's contingent choices maximize her conditional preferences, every consistent plan will be (ex ante) optimal with respect to her static preferences, if and only if each event in the domain of her prior and its complement constitute a decomposable split.

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1 Introduction

Consider a decision-maker (hereafter, DM) operating in a Savage setting of subjective uncertainty described by a state space Ω, in which any subset E of Ω is referred to as an event with its complement denoted by $\Omega \setminus E$. An object of choice for our DM is an uncertain prospect or *act*, formally identified as a mapping f in which for each state ω in Ω, $f(\omega)$ is the outcome from some set X that is associated with the act when the state ω is realized. Following Machina and Schmeidler (1992) we assume ever act is *simple* in the sense that its image is a finite subset of X. And in line with Gul and Pesendorfer (2014) we shall take X to be an interval of the real line.[1]

In the tradition of the voluminous literature initiated by Ellsberg (1961), our DM perceives there to be ambiguity arising as a result of her possessing only *incomplete information* about the underlying stochastic process that determines the resolution of the uncertainty she faces. In particular, this means she is not comfortable quantifying with a precise probability the uncertainty she associates with each and every event. However, in common with Gul and Pesendorfer's (2014) family of expected uncertain utility (EUU) maximizers, there exists a rich collection of events she deems *measurable* over which a probability, μ, that we refer to as her *prior*, may be defined.

In Gul and Pesendorfer's theory, an EUU maximizer deems measurable any event for which both it and its complement satisfy Savage's (1954) postulate **P2**.[2] Gul and Pesendorfer refer to any such event as *ideal* and to contrast that with the alternative notion we introduce below, we shall refer to any ideal event and its complement as constituting an *ideal split* of the state space. To express this formally, for any pair of acts f and g and any event E, let $f_E g$ denote the act that agrees with f on E and with g on $\Omega \setminus E$, and let $\succsim$ denote her (weak) preference relation defined over acts, with $\succ$ and $\sim$ denoting its asymmetric and symmetric parts, respectively. We say the event B is ideal (or equivalently, the two-element partition $\{B\,, \Omega \setminus B\}$ constitutes an ideal split of the state space), if for any pair of acts f and g:

$$g_B f \succsim f \implies g \succsim f_B g \quad \text{and} \quad g_{\Omega\setminus B} f \succsim f \implies g \succsim f_{\Omega\setminus B}\, g\,.$$

Savage motivates **P2** as the natural way to operatonalize his *sure-thing principle* which can (informally) be expressed as follows:

> *If the DM weakly prefers g to f, either knowing that the event B obtains, or knowing that the complement of the event B obtains, then she weakly prefers g to f.*

[1] As Grant et al. (1992) demonstrate in the context of choice under risk (where probabilities are objective and known by the DM), assuming the outcome set is a one-dimensional totally ordered set may be viewed as essentially without much loss of generality, if, for example, this subset of the real line corresponds to a (money-metric) *indirect utility* (for given prices) derived from the DM's preferences over an underlying multi-dimensional consumption set.

[2] Although this differs from the formal definition given in Gul and Pesendorfer (2014, p3), their Lemma B0 (p25) establishes the equivalence of their definition and the one presented here.

Since objects like "g if the event B obtains" are at best subacts, Savage needs to extend these subacts to the entire state space, which he does as follows:

> *What technical interpretation can be attached to the idea that g would be preferred to f if B were known to obtain? Under any reasonable interpretation, the matter would seem not to depend on the values f and g assume at states outside of B. There is, then, no loss of generality in supposing that f and g* ***agree*** *with each other except in B; ... . The first part of the sure-thing principle can now be interpreted thus: If, after being modified so as to agree with one another outside of B, f is not preferred to g; then f would not be preferred to g, if B were known.* Savage (1972, p22). (Boldface text in the original.)

To see how this "technical interpretation" can "operationalize" Savage's sure-thing principle, first assume the usual maintained property that the (unconditional) preference $\succsim$ is complete and transitive (that is, it satisfies Savage's **P1**). Next, if for each ideal event B, we define $\succsim_B$ to be the relation derived from $\succsim$ by setting $g \succsim_B f$, whenever $g_B f \succsim f$, it is straightforward to check that the derived relation $\succsim_B$ is *consequentialist* and inherits the properties of completeness and transitivity.[3] By interpreting $\succsim_B$ (respectively, $\succsim_{\Omega \setminus B}$) as the DM's preference conditional on knowing B (respectively, the complement of B) obtains, ideal splits are precisely those two-element partitions of the state space for which the corresponding pair of conditional preference relations $\succsim_B$ and $\succsim_{\Omega \setminus B}$ are both consequentialist and, together with the original (unconditional) relation $\succsim$, exhibit Skiadas (1997) *coherence* a consistency condition for conditional preferences that requires for every pair of acts f and g:

$$g \succsim_B f \text{ and } g \succsim_{\Omega \setminus B} f \implies g \succsim f\,.$$

To see why, first notice that the two conditional preference statements $g \succsim_B f$ and $g \succsim_{\Omega \setminus B} f$ entail (by definition) that $g_B f \succsim f$ and $f_B g \succsim f$, respectively. And since B is ideal, $g_B f \succsim f$ implies $g \succsim f_B g$. Hence from the transitivity of $\succsim$ we have $g \succsim f$, as required.

Our point of departure from Gul and Pesendorfer stems from a discussion in Grant et al. (2000) questioning how "innocuous" is Savage's technical interpretation. To motivate as well as illustrate the point they wish to make, they ask the reader to consider three acts, "to buy (a new house)," "to stay put", and "to emigrate to Japan." They note that Savage interprets the statement "the DM would prefer to buy a new house over staying put if she knew event B obtains" as implying that the agent prefers the *act*, "to buy if (the event) B obtains and to emigrate to Japan otherwise," over the *act* "to stay put if B obtains and to emigrate to Japan otherwise." Moreover, Savage requires the same preference to hold if "to emigrate to Japan" were changed to any other activity, irrespective of its outcomes. However, as they highlight,

> *"it is precisely this separability – preference between a pair of acts on the event B do not*

[3] The relation $\succsim_B$ is consequentialist in the sense that the preference between any pair of acts f and g only depends on those outcomes that can arise for f and g in states that lie in the event B (or equivalently, is not affected by what outcomes might have arisen for either f or g in states that are not in B).

depend on what happens off B – that has been challenged on experimental and introspective grounds." Grant et al. (2000, p171).

They posit it is possible that the DM's attitude toward owning and living in a new house might differ if she knew it came in place of a life in Japan. For example, compared with the excitement of embarking on a new life in Japan, the new house might seem small and confining. Alternatively, compared to typical Japanese real estate, the new house might seem large and more attractive.

So instead they suggest the statement "the DM would prefer to buy a new house over staying put if she knew B obtains" need only imply that the agent prefers the act, "to buy if B obtains and to stay put otherwise," over the act "to stay put," thus eschewing the need to draw inferences about her preferences among esoteric acts involving her emigrating to Japan should B not obtain. Likewise, the statement "the DM would prefer to buy over staying put if she knew B does not obtain" need only imply that the agent prefers the act, "to buy if B does not obtain, and to stay put otherwise," over the act "to stay put." The axiom they propose as an alternative to **P2** is the *weak decomposability property*. It simply requires if in both comparisons the former act is preferred to the latter, then the DM prefers the act "to buy" over the act "to stay put." A similar argument against **P2** is made in Tversky and Shafir (1992), which also contains empirical data on this issue.

Accepting these arguments, we propose our DM views as measurable any event for which it and its complement constitute what we dub a *decomposable split* of the state space. Formally, a two-element partition, $\{R, \Omega \setminus R\}$, constitutes an *decomposable split*, if for every pair of acts f and g:

$$g_R f \succ f \text{ and } f_R g \succ f \quad \text{implies} \quad g \succ f\,.$$

That is, starting off from the act f, if the DM is made better off by replacing f with g on R and she is also made better off by replacing g with f on the complement of R then she is (unconditionally) made better off by switching from f to g.

One might object that our definition of a decomposable split gives a privileged "status-quo" role to the act f, that corresponds to the act "to stay put" in the example above. Recall, we interpreted the statement "the DM would prefer to buy over staying put if she knew B obtains" to mean the entrepreneur is expressing a (strict) preference for the act "to buy if B obtains and to stay put otherwise," over the act "to stay put." An equally reasonable interpretation is that she prefers the act "to buy" over the act "to stay put if B obtains, and to buy otherwise." However, if the DM's preference relation $\succsim$ satisfies Savage's postulate **P1**, that is, it is complete and transitive, then we show in the Appendix 6 (see Lemma 6(i)) for any decomposable split $\{R\,,\Omega\backslash R\}$, the following decomposability property in the *reverse* direction also holds for any pair of acts f and g:

$$g \succ f_R g \text{ and } g \succ g_R f \quad \text{implies } g \succ f\,.$$

That is, starting from the act g, if the DM is made worse off replacing g with f on R and she is also made worse off by replacing g with f on $\Omega \backslash R$ then she is (unconditionally) made worse off by switching from g to f.

A second objection to our definition is that the derived conditional orderings are no longer consequentialist. However, foreshadowing the results from Section 5 in which we extend our analysis to a dynamic setting, we show that the form of non-consequentialism arising from our generalization to decomposable events is compatible with dynamically consistent choice.

We take the domain of our DM's prior μ to comprise precisely those events for which they and their respective complements constitute decomposable splits. We refer to any event in the domain of μ as either decomposable or measurable. Correspondingly, we refer to any act that is measurable with respect to μ as either decomposable or measurable.

For any decomposable act, the implicit separability embodied in the property of weak decomposability means that the act's *certainty equivalent*, $V(g) \in X$ admits an *implicit linear utility representation* characterized as the (unique) solution to:

$$\sum_{x \in X} U(x, V(g))\, \mu(g^{-1}(\{x\})) = 0\,, \tag{1}$$

in which $U(\cdot\,,\cdot)$ is a *balanced utility*: a continuous function that is increasing in its first argument, decreasing in its second, and normalized so that $U(x,x) = 0$, for all $x \in X$.

Remark 1 *If we require every event in the domain of μ to be ideal then $U(x,c) = v(x) - v(c)$ for some* Bernoulli utility $v\colon X \to \mathbb{R}$. *Moreover, by rearranging expression (1), we obtain the following explicit expression for $V(g)$:*

$$V(g) = v^{-1}\left(\sum_{x \in X} v(x)\, \mu(g^{-1}(\{x\}))\right)\,, \tag{2}$$

So in this case the restriction of the DM's preferences to measurable acts conforms to subjective expected utility theory.

To model her valuation of a general act f, we adapt Gul and Pesendorfer's construction that assigns to f, a measurable *envelope*, $\mathbf{f}$. We keep their notion of a *measurable partition* of the state space induced by the inverse image of f.[4] Each element of this partition corresponds to a (finite) subset Y of outcomes. The essential difference is that whereas they define an envelope as a mapping from states to intervals of outcomes, defined in terms of minimum and maximum outcomes in Y, we retain the entire set Y. That is, the envelope $\mathbf{f}$ maps to Y precisely those states in the element of the measurable partition induced by the inverse image of f that corresponds to Y. Analogous to

[4]Gul and Pesendorfer refer to such a partition as an *ideal* "split," but in this paper we reserve the use of the term "split" to refer specifically to two-element partitions of the state space.

the (interval) envelopes in Gul and Pesendorfer (2014), $\mathbf{f}$ is characterized as the minimal measurable mapping from states to outcome-sets that contains f.

In effect, the interpretation of envelopes in terms of belief and plausability functions, as described in Gul and Pesendorfer (2014), becomes even more straightforward: the belief in the outcome set Y obtaining is the total probability assigned to Y and all its subsets in the envelope, while the plausability of Y is the total probability of all subsets containing at least one element of Y. Moreover, the prior μ and the envelope $\mathbf{f}$, induces the outcome-set lottery in which for each finite set of outcomes Y, the probability assigned to Y is given by $\mu\left(\mathbf{f}^{-1}(Y)\right)$.

The axioms we adopt guarantee existence and uniqueness of envelopes. Conversely, the richness of the state space will ensure that for any outcome set lottery L, there exists a corresponding act f with an associated envelope $\mathbf{f}$, for which $L = \mu \circ \mathbf{f}^{-1}$. Imposing now that the DM is indifferent among acts inducing the same outcome-set lottery, this leads to the representation of preferences as in (2), with outcome x replaced by a (finite) set of outcomes Y, and the act f replaced by its envelope $\mathbf{f}$. That is, the certainty equivalent $V(f)$ can be characterized as the unique solution to

$$\sum_{Y \subset X, |Y| < \infty, Y \neq \emptyset} U(Y, V(f))\, \mu\left(\mathbf{f}^{-1}(Y)\right) = 0\,. \tag{3}$$

Thus our DM's preference relation, $\succsim$, is characterized by the pair $\langle \mu, U \rangle$, comprising her prior μ and and her balanced *outcome-set* utility, U. This provides us with a clean separation of the ambiguity she perceives to be present given her knowledge about the random process governing the resolution of the uncertainty she faces from her attitude toward risk (that is, measurable uncertainty). The former is characterized by those events that lie outside the domain of her prior while the restriction of her balanced outcome-set utility to singleton outcome-sets encodes the latter. Finally, her attitude toward (general) uncertainty involving both risk and ambiguity is embodied in her (unrestricted) balanced outcome-set utility.

We dub the class of preference relations that admit such a representation as the family of *Implicit Linear Uncertain Utility (ILUU) maximizers*. We develop the formal definition of an ILUU maximizer in section 2 and then in section 3 we provide five special subclasses. For the first three, the restriction of the preference relation to decomposable acts conforms to expected utility (and so, every decomposable split is actually an ideal split) but for the last two this restriction need only conform to Chew's (1983) and Dekel's (1986) betweenness property and thus can accommodate Allais style violations of expected utility. Section 4 contains our axiomatic characterization.

In section 5 we consider a dynamic setting in which a DM with rich coherent beliefs represented by a prior μ is tasked with choosing an act from a given (finite) menu conditional on a signal.

Although this DM need not necessarily be an ILUU maximizer, consider first the case where the events in the domain of her prior are ideal, as, for example, is the case for an EUU maximizer.

Recall from above the definition of the preference $\succsim_B$ conditional on a DM knowing the ideal event B obtains, is by construction consequentialist. Moreover, as Savage notes:

> *"It is possible and instructive to give an atemporal analysis of the following temporally described situation: The person must decide between f and g after he finds out, that is, observes, whether B obtains ... "* Savage (1972, p23)

Suppose it is the case that both $g \succsim_B f$ and $g \succsim_{\Omega\setminus B} f$, then having initially adopted the plan of action $g_B g$ $(\equiv g)$ the DM will not have a strict incentive to deviate from it irrespective of whether she learns B has or has not obtained. That is, $g_B g$ is a *consistent plan* (of action) in the sense of Strotz (1955) and Siniscalchi (2011). Moreover, it follows from $\succsim_B$ and $\succsim_{\Omega\setminus B}$ being consequentialist and $\succsim$ being transitive, that $g_B g$ is also ex ante optimal. That is, $g \succsim \widehat{f}$ for all $\widehat{f}$ in $\{f\,, g_B f\,, f_B g\,, g\}$, the set of available plans contingent on the DM learning whether B has or has not obtained.[5]

This insight extends to the case where the DM makes her choice from a given finite set of acts after observing which element of a finite partition of ideal events obtains. Namely, any consistent plan, that is one from which the DM never has a strict incentive to deviate, no matter which element of the partition obtains, must be optimal with respect to her static preferences.[6]

We investigate to what extent it is possible to drop consequentialism but retain this form of dynamic consistency with the proviso that the DM's perception of a signal's realizations is adapted to events that she can measure with her prior. In particular, this will entail for each possible realization of the signal, the corresponding event on which the DM conditions her preferences will be "measurable". In other words, we require any information the DM gleans from a signal must be measurable with respect to the DM's prior and so accord with her partial knowledge of the stochastic process that resolves the uncertainty she faces.

Our key result is that all consistent plans are ex ante optimal, if and only if each measurable event and its complement constitute a decomposable split. Roughly speaking, this means that the only type of DM with rich coherent beliefs for whom all consistent plans are ex ante optimal comes from the class of ILUU maximizers.

Unless stated otherwise, all proofs appear in the appendix.

2 The Model

As already noted above in the introduction, we operate in a setting of purely subjective uncertainty described by a state space Ω. The objects of choice are acts that for each state of nature $\omega \in \Omega$,

[5]To see this, as $\succsim_B$ (respectively, $\succsim_{\Omega\setminus B}$) is consequentialist, $g \succsim_B f$ (respectively, $g \succsim_{\Omega\setminus B} f$) implies $g \succsim f_B g$ and $g_B f \succsim f$ (respectively, $g \succsim g_B f$ and $f_B g \succsim f$). $g \succsim f$ then follows from transitivity.

[6]A formal treatment along with an axiomatization of subjective expected utility and Bayesian updating in a conditional decision problem can be found in Ghirardato (2002). Notice this is a setting in which *every* event is assumed to be ideal.

deliver an outcome x from a set X that we take to be a compact interval $[\ell, m]$ in $\mathbb{R}$. Each act f is simple, that is, its image $f(\Omega)$ is a finite subset of X.

We denote the set of all acts by F. We identify any outcome $x \in X$ with the (constant) act f in which $f(\omega) = x$ for all ω. And with further (albeit fairly standard) abuse of notation, X will also refer to the set of constant acts.

For any pair of events $E, B \subseteq \Omega$, $B \backslash E$ shall denote the set of elements that are in B but not in E. For any pair of acts f and g in F and any event $E \subset \Omega$, we write $f_E g$ for the act that agrees with f on E and with g on $\Omega \backslash E$.

Throughout we will employ the following convention with our notation. For any arbitrary set K, $\mathcal{K}$ (that is, calligraphic K) shall denote the set of all non-empty finite subsets of K. And for each $J \in \mathcal{K}$, $\mathcal{K}^J$ shall denote the set of non-empty subsets of J (that is, $\mathcal{K}^J = \{J' \in \mathcal{K} \colon J' \subseteq J\}$). So for example, $\mathcal{X}$ (respectively, $\mathcal{F}$) denotes the set of all finite subsets of outcomes (respectively, acts).[7]

The DM is characterized by her preferences over acts, a binary relation $\succsim$ on F.

We begin our description of an Implicit Linear Uncertain Utility (ILUU) maximizer by first noting she possesses *rich coherent beliefs*. As we outlined in the introduction, this entails the existence of a sufficiently rich collection of (*risky*) events, constituting a σ-algebra of subsets of Ω, over which can be defined a countably-additive and convex-ranged probability measure μ (her 'prior') with which the DM can *quantify precisely* the uncertainty she associates with each risky event. We denote the domain of μ by $\mathbf{R}$.

Countable-additivity requires the probability of the union of a countable collection of disjoint measurable events from $\mathbf{R}$ equals the infinite sum of the probabilities of these events. For μ to be convex-ranged requires for any event R in $\mathbf{R}$ and any r in $(0,1)$ there exists a subset $B \subset R$ that is in $\mathbf{R}$ and for which $\mu(B) = r\mu(R)$.

As we noted in the introduction, there is a natural way to use this prior to identify with each act its *outcome-set* envelope. Let F_μ be the set of measurable acts and let $\mathbf{F}_\mu$ be the set of measurable functions $\mathbf{f} \colon \Omega \to \mathcal{X}$ with finite range. We refer to elements of $\mathbf{F}_\mu$ as outcome-set acts.

Definition 1 (Envelope of an act) *The outcome-set act* $\mathbf{f} \in \mathbf{F}_\mu$ *is the* envelope *of* f *if*

(*i*) $f(\omega) \in \mathbf{f}(\omega)$ *for all* $\omega \in \Omega$, *and*

(*ii*) *for any* $\mathbf{g} \in \mathbf{F}_\mu$:

$$f(\omega) \in \mathbf{g}(\omega) \quad \textit{for all } \omega \in \Omega \implies \mu\left(\{\omega \in \Omega \colon \mathbf{f}(\omega) \subseteq \mathbf{g}(\omega)\}\right) = 1\,.$$

Remark 2 *It readily follows from Definition 1 that for any event* R *in* $\mathbf{R}$ *and any pair of acts* f *and* g *with respective envelopes* $\mathbf{f}$ *and* $\mathbf{g}$, $\mathbf{f}_R\mathbf{g}$ *is the envelope of* $f_R g$.

[7] And for any act f in F, $\mathcal{X}^{f(\Omega)}$ denotes the set of all finite non-empty subsets of outcomes in the image of f.

To see how to construct the envelope of an act, first consider the *inner measure* of μ, denoted by μ_*, that is derived from the prior by assigning to each event $E \subset \Omega$ the weight $\mu_*(E) \in [0,1]$ that is the solution to:

$$\sup_{R \in \mathbf{R}, R \subseteq E} \mu(R) \ .$$

Since μ is countably additive, the supremum is attained. We shall refer to the measurable event $\underline{E} \in \mathbf{R}$, as the *inner-sleeve* of E, if $\underline{E} \subseteq E$ and $\mu(\underline{E}) = \mu_*(E)$. The *outer-sleeve*, denoted by $\overline{E}$, is a measurable event in $\mathbf{R}$, for which $E \subseteq \overline{E}$ and $\mu(\overline{E}) = 1 - \mu_*(\Omega \backslash E)$.[8]

Using the inner measure we can associate with any mapping from the state space with a finite range a measurable partition of the state space that is generated by the mapping's inverse image.

Definition 2 (Measurable partitions) *The* measurable partition *(of the state space) associated with a mapping $\phi : \Omega \to K$ and denoted by $\{R_\phi^J \in \mathbf{R} : J \in \mathcal{K}^{\phi(\Omega)}\}$, is inductively defined as follows:*

1. *For each element $k \in \phi(\Omega)$, set $R_\phi^{\{k\}} := \underline{\phi^{-1}(k)}$.*
2. *For each $J \in \mathcal{K}^{\phi(\Omega)}$ such that $|J| > 1$, set*

$$R_\phi^J := \underline{\phi^{-1}(J)} \backslash \left(\bigcup_{\widehat{J} \in \mathcal{K}^J, \widehat{J} \neq J} R_\phi^{\widehat{J}} \right) .$$

*We refer to R_ϕ^J as the ϕ-*marginal inner-sleeve *of the set J.*

Now consider an arbitrary act f in F and its associated measurable partition $\{R_f^Y \in \mathbf{R} : Y \in \mathcal{X}^{f(\Omega)}\}$. Notice it follows from Definition 2 that for each set of outcomes $Y \in \mathcal{X}^{f(\Omega)}$ we have,

$$\bigcup_{\widehat{Y} \in \mathcal{X}^Y} R_f^{\widehat{Y}} \subseteq f^{-1}(Y) \text{ and } \mu_*\left(f^{-1}(Y)\right) = \mu\left(\underline{f^{-1}(Y)}\right) = \sum_{\widehat{Y} \in \mathcal{X}^Y} \mu\left(R_f^{\widehat{Y}}\right) .$$

That is, viewing $\mu_*(f^{-1}(Y))$ as the *belief* the act f will result in an outcome from the set Y obtaining, we see it equals the total probability assigned to the f-marginal inner-sleeve of Y and the f-marginal inner-sleeves of all its subsets.

Remark 3 *It readily follows from Definition 1 that for any act $f \in F$ with associated measurable partition $\{R_f^Y \in \mathbf{R} : Y \in \mathcal{X}^{f(\Omega)}\}$, the outcome-set act $\mathbf{f} \in \mathbf{F}_\mu$ in which $\mathbf{f}(\omega) = Y$ whenever $f(\omega) \in R_f^Y$ is the (essentially unique) envelope of f.*

An *outcome-set lottery* is a finite ranged function $L : \mathcal{X} \to [0,1]$ satisfying $\sum_{Y \in \mathcal{X}} L(Y) = 1$. Let $\Delta(\mathcal{X})$ denote the set of (simple) outcome-set lotteries defined on $\mathcal{X}$, and for each $Y \in \mathcal{X}$, let δ_Y denote the degenerate outcome-set lottery that assigns probability one to the outcome-set Y, that

[8] Notice that the inner- and outer- sleeves are unique up to a set of μ-measure 0.

is, $\delta_Y(Y) = 1$ and $\delta_Y(Z) = 0$ for all $Z \neq Y$. We associate with the act f the outcome set lottery $\mu \circ \mathbf{f}^{-1}$.

Recalling the approach of Dempster (1967) and Shafer (1976), we shall interpret the outcome-set lottery $\mu \circ \mathbf{f}^{-1}$ as encoding how the DM weights that part of the evidence supporting the belief that the act f leads to an outcome in a given set of outcomes obtaining that is not well-specified enough to allow her to distribute any of it across any of the elements of that set or any of the other strict subsets of that set of outcomes. In other words, for each Y in $\mathcal{X}^{f(\Omega)}$ we shall interpret $\mu \circ \mathbf{f}^{-1}(Y) \left(= \mu\left(R_f^Y\right)\right)$ as the weight assigned by the DM to evidence that directly supports the act f leading to an outcome in Y obtaining that cannot be further refined in terms of any of the strict subsets of Y.

Analogous to Grant's (1995 p163) rendition of Machina and Schmeidler's (1992) concept of probabilistic sophistication, we require that no relevant preference information is lost by this association.

Definition 3 (Coherent Beliefs) *The prior* μ *is a* coherent belief *for the preference relation* $\succsim$, *if for each pair of acts* f *and* $\widehat{f}$, *with respective envelopes* $\mathbf{f}$ *and* $\widehat{\mathbf{f}}$, $f \sim \widehat{f}$ *whenever* $\mu \circ \mathbf{f}^{-1} = \mu \circ \widehat{\mathbf{f}}^{-1}$.

Remark 4 *We assume that differences between acts with same envelope are either irrelevant, or even unknown to the DM. In particular, we do not exclude the DM preferring* f *to* g *with* $f \ll g$, *when this inequality cannot be distilled from their envelopes. So, whereas we adopt probabilistic sophistication in terms of envelopes as a standard behavioral assumption, the identification of acts with their envelope in our framework is primarily on epistemological grounds.*

One more element is needed to define an ILUU maximizer, a *balanced outcome-set utility* U that specifies the utility $U(Y, c)$ of a outcome-set Y in a lottery of value c, the balance point. The monotonicity property we impose relies on the following concept of outcome-set dominance.

Definition 4 (Outcome-Set Dominance) *We say an outcome set* $Y = \{y_1, \ldots, y_k\}$, *with* $y_i > y_{i+1}$ *for all* $i = 1, \ldots, k-1$, dominates *outcome set* $Z = \{z_1, \ldots, z_n\}$, *with* $z_j > z_{j+1}$ *for all* $j = 1, \ldots, n-1$, *whenever either*: $k \leqslant n$ *and* $y_i > z_i$ *for each* $i = 1, \ldots, k$; *or,* $k > n$ *and* $y_{j+(k-n)} > z_j$ *for each* $j = 1, \ldots, n$. *For* $k, n > 1$, *the required inequalities involving extreme outcomes* ℓ *or* m *may be non-strict.*

A balanced outcome-set utility $U\colon \mathcal{X} \times X \to \mathbb{R}$ satisfies the following four properties.

(i) *Balance Point Normalization*: for any $c \in X$: $U(\{c\}, c) = 0$.

(ii) *Balance Point Monotonicity*: U is decreasing and continuous in its second argument.

(iii) *Outcome-Set Monotonicity:* for any $c \in X^0$, $U(Y, c) > U(Z, c)$, whenever Y dominates Z.

(iv) *Outcome-Set Continuity:* for any $Y \in \mathcal{X}$ and any sequence of outcome-sets $Z^n \in \mathcal{X}$ with $|Z^n| = |Y|$ for all n, if Z^n converges pointwise to Y, then $U(Z^n, c) \to U(Y, c)$ for all c.

Remark 5 *The continuity property (iv) guarantees that the non-strict version of monotonicity (iii) also holds true in the case Y* weakly *dominates Z, defined as the non-strict variant of Definition 4.*

We have now assembled all the parts we need to define a DM who is an ILUU maximizer.

Definition 5 (The Class of ILUU Maximizers) *A preference relation $\succsim$ is a member of the class of* Implicit Linear Uncertain Utility maximizers *if there exists a prior μ and a balanced outcome-set utility U, such that $\succsim$ admits a certainty equivalent representation $V\colon F \to X$, in which $V(f)$ is the (unique) solution to*

$$\sum_{Y \in \mathcal{X}^{f(\Omega)}} U(Y\,, V(f))\, \mu\left(\mathbf{f}^{-1}\,(Y)\right) = 0\,, \tag{4}$$

where $\mathbf{f}$ *is the envelope of f.*

3 Examples of ILUU Maximization

We present five subclasses of ILUU maximizers. The first two are the model of EUU maximization from Gul and Pesendorfer (2014) and its special subclass of Hurwicz expected utility (HEU) maximization from Gul and Pesendorfer (2015) that were developed in (essentially) the same setting as this paper. The third is a subjective version of a preference model over *belief functions* introduced by Eichberger and Pasichinichenko (2021). All three can be viewed as subclasses of the general preference class of linear utility over belief functions as introduced and axiomatized in Jaffray (1989). All three share the feature that the restriction of the preference relation to decomposable acts conforms to expected utility theory, and indeed for these three subclasses any decomposable split of the state space is also an ideal split which is reflected in the additive separability of the associated balanced outcome-set utility.

For the fourth and fifth, however, their associated balanced outcome-set utility need not be additively separable and so the restriction of preferences to decomposable acts need not conform to expected utility. Instead they need only satisfy the betweenness property of Chew (1983) and Dekel (1986). In particular, the fourth (respectively, the fifth) may be viewed as a subjective uncertainty analogue to Chew's (1983) weighted expected utility (respectively, Gul's (1991) disappointment averse) model of preferences over lotteries (with objectively given probabilities). The fifth subclass is particularly popular for use in applications as its representation is only one parameter richer than expected utility which is reflected in the specification of its associated balanced utility below.

1. EUU maximization: there exists an interval utility

$$u : \{[y\,, z] \in X \times X : y \leqslant z\} \to \mathbb{R}\,,$$

that is continuous and monotonic in the sense that

$$u(y\,,z) > u(y'\,,z') \text{ whenever } y > y' \text{ and } z > z'\,,$$

such that

$$U(Y,c) = u\left(\min_{y\in Y} y\,,\max_{z\in Y} z\right) - u(c,c)\,.$$

In the other four examples $v : X \to \mathbb{R}$ denotes a Bernoulli utility that is continuous and monotonic in the sense that $v(x) > v(y)$, whenever $x > y$.

2. HEU maximization: there exists an $\alpha \in [0,1]$ such that

$$U(Y,c) = \alpha v\left(\min_{y\in Y} y\right) + (1-\alpha)\, v\left(\max_{z\in Y} z\right) - v(c)\,.$$

3. Quasi-Arithmetic Mean Uncertain Utility (QAMUU) maximization: there exists a (second-order) utility $\phi\colon v(X) \to \mathbb{R}$ that is continuous and monotonic, in the sense that $\phi(w) > \phi(w')$ whenever $w > w'$, such that

$$U(Y,c) = \phi^{-1}\left(\frac{1}{|Y|}\sum_{x\in Y} \phi\left(v(x)\right)\right) - v(c)\,.$$

For the last two examples $e\colon \mathcal{X} \to X$ denotes an *unambiguous equivalent* that satisfies set-monotonicity and set-continuity.

4. Weighted Uncertain Utility (WUU) maximization: there exists a *utility weighting* $w\colon X \to \mathbb{R}_{++}$ such that

$$U\left(Y,c\right) = w\left(e(Y)\right)\left(v(e(Y)) - v(c)\right)\,. \tag{5}$$

5. Disappointment Averse Uncertain Utility (DAUU) maximization: there exists a $\beta > -1$, such that

$$U\left(Y,c\right) = \begin{cases} v\left(e(Y)\right) - v\left(c\right) & \text{if } e(Y) \leqslant c \\ \frac{v(e(Y))-v(c)}{1+\beta} & \text{if } e(Y) > c \end{cases}\,. \tag{6}$$

For both EUU and HEU maximizers, the balanced utility of the outcome set Y for a given c only depends on its worst and best elements.[9] For the third, as its name suggests, the balanced utility of the outcome set Y for a given c is the quasi-arithmetic mean of the Bernoulli utilities associated with that set, taken with respect to the second-order utility ϕ, less the Bernoulli utility of c. Eichberger and Pasichinichenko argue this embodies a *principle of insufficient reason* for evaluating ambiguous

[9] For these classes it is understood that they belong to ILUU on every closed subset of X^0. The dominance criterion for the extreme outcomes ℓ, m is not satisfied.

sets of possible outcomes. For each of the three, standard definitions of aversion to ambiguity from the literature correspond to $u(y,z) \leqslant (u(y,y)+u(z,z))/2$, $\alpha \leqslant 1/2$, and ϕ is concave, respectively.

It is straightforward to show that the property of additive separability for the associated balanced outcome-set utilities of the first three, means their corresponding implicit representations as given in expression (4) can be rearranged to provide explicit representations.

The fourth also readily yields an explicit representation. To see this, notice that for a WUU maximizer, substituting (5) into (4), it follows that the certainty equivalent $V(f)$ can be expressed as the solution to

$$\sum_{Y \in \mathcal{X}} w\left(e\left(Y\right)\right)\left[v\left(e\left(Y\right)\right) - v\left(V\left(f\right)\right)\right]\mu\left(\mathbf{f}^{-1}\left(Y\right)\right) = 0\,.$$

A simple rearrangement yields

$$V\left(f\right) = v^{-1}\left(\frac{\sum_{Y\in\mathcal{X}} w\left(e\left(Y\right)\right)v\left(e\left(Y\right)\right)\mu\left(\mathbf{f}^{-1}\left(Y\right)\right)}{\sum_{Y\in\mathcal{X}} w\left(e\left(Y\right)\right)\mu\left(\mathbf{f}^{-1}\left(Y\right)\right)}\right)\,.$$

As we already noted above, in the fifth subclass the risk preferences may be seen as conforming to Gul's (1991) risk preference model of *disappointment aversion*. Gul highlights a key feature of disappointment averse preferences is that they admit a representation that is only one parameter richer than expected utility. This corresponds to the parameter $\beta > -1$ in expression (6). Expected utility is the special case in which $\beta = 0$. Risk aversion corresponds to the concavity of the Bernoulli utility v and $\beta \geqslant 0$. The property of disappointment aversion which Gul shows is both necessary and sufficient to generate Allais style choice patterns, requires $\beta > 0$. A DM with $\beta < 0$ is referred to as elation seeking.

The property required for a DAUU maxmizer (as well as a WUU maximizer) to exhibit aversion to ambiguity depends on the particular specification of the unambiguous equivalent $e(Y)$. For example if it is implicitly defined as the solution to

$$u\left(e(Y)\,, e(Y)\right) = u\left(\min_{y\in Y} y\,, \max_{z\in Y} z\right)\,,$$

where $u(\cdot\,,\cdot)$ is an interval utility with same properties as one that appears in the representation of an EUU maximizer, then aversion to ambiguity corresponds to $u(y,z) \leqslant (u(y,y)+u(z,z))/2$. Alternatively, if the unambiguous equivalent takes the form

$$e(Y) = v^{-1}\circ\phi^{-1}\left(\frac{1}{|Y|}\sum_{x\in Y}\phi\left(v(x)\right)\right)\,,$$

then aversion to ambiguity corresponds to ϕ being concave.

Although not as immediately apparent, the preferences of a DAUU maximizer also admit an explicit representation as well. To see how this can be achieved, first following Cerreia-Vioglio et al. (2020), let $v_w(\cdot)$ denote the "conditional" Bernoulli utility obtained by setting

$$v_w(x) = \begin{cases} v(x) & \text{if } v(x) \leqslant w \\ \frac{v(x)+\beta w}{1+\beta} & \text{if } v(x) > w \end{cases} . \tag{7}$$

Comparing expressions (6) and (7) notice that

$$U\left(Y, v^{-1}(w)\right) \equiv v_w\left(e(Y)\right) - w\,.$$

A straightforward application of Cerreia-Vioglio et al.'s (2020) Theorem 3 (p1514) yields:

(i) if $\beta > 0$ (that is, risk preferences exhibit disappointment aversion) then

$$V(f) = \min_{w \in v(X)} v_w^{-1}\left(\sum_{Y \in \mathcal{X}} v_w\left(e(Y)\right)\mu\left(\mathbf{f}^{-1}(Y)\right)\right) :$$

(ii) if $\beta = 0$ (that is, risk preferences conform to expected utility theory) then

$$V(f) = v^{-1}\left(\sum_{Y \in \mathcal{X}} v\left(e(Y)\right)\,\mu\left(\mathbf{f}^{-1}(Y)\right)\right) :$$

(iii) if $\beta < 0$ (that is, risk preferences exhibit elation seeking) then

$$V(f) = \max_{w \in v(X)} v_w^{-1}\left(\sum_{Y \in \mathcal{X}} v_w\left(e(Y)\right)\mu\left(\mathbf{f}^{-1}(Y)\right)\right) .$$

Before turning to the characterization of the class of ILUU maximizers, we present the following example that illustrates how the representation of an QAMUU maximizer above can fail to satisfy uniform continuity.

Example 1 *Take the state space to be* $\Omega := [0\,,1] \times [0\,,1]$ *with generic element denoted by* $(\omega_1\,,\omega_2)$ *and take the outcome set to be* $X := [0\,,1]$. *The DM is a member of the class of QAMUU maximizers. Her prior* μ *is defined on the set of events* $\mathbf{R} := \{A \subseteq \Omega\colon E = B \times [0\,,1],\ \textit{for some } B \in \mathcal{L}\}$, *where* $\mathcal{L}$ *denotes the set of Lebesgue measurable subsets of* $[0\,,1]$. *For each* $B \in \mathcal{L}$, $\mu(B \times [0\,,1])$ *is equal to the Lebesgue measure of* B. *Her balanced outcome-set utility is given by*

$$U(Y, c) := \frac{\sum_{x \in Y} x}{|Y|} - c\,.$$

Now consider the sequence of acts $\{f^n\}_{n=1}^{\infty}$ *in which*

$$f^n(\omega_1,\omega_2) = \begin{cases} 0 & \textit{if } \omega_2 \geqslant 2/3 \\ 1/n & \textit{if } \omega_2 \in [1/3, 2/3) \\ 1 & \textit{if } \omega_2 \in [0, 1/3) \end{cases}$$

Notice first that the sequence converges uniformly to the act f *given by*

$$f(\omega_1,\omega_2) = \begin{cases} 0 & \textit{if } \omega_2 \geqslant 1/3 \\ 1 & \textit{if } \omega_2 < 1/3 \end{cases} .$$

Now for each act f^n *in this sequence and for each outcome set* Y *that is a non-empty strict subset of* $\{0\,,1/n\,,1\}$ $(= f^n(\Omega))$, *we have* $\mu_*\left((f^n)^{-1}(Y)\right) = 0$. *Hence it follows that* $V(f^n) = 1/(3n) + 1/3$. *Thus,*

$$\lim_{n\to\infty} V(f^n) = \frac{1}{3} \neq \frac{1}{2} = V(f)\,,$$

a violation of uniform continuity.

As a consequence, so that we can include the subclass of QAMUU maximizers within the family of ILUU maximizers, we shall appropriately modify the notion of continuity we require the preferences to satisfy in the axiomatic characterization presented in the next section.

4 Characterization

We begin our characterization of the class of ILUU maximizers by first specifying what property an event and its complement must satisfy in order for us to infer that the DM deems them to be ‘risky’ and thus ‘measurable’, thereby lending themselves to a precise quantification by her prior of the uncertainty she associates with them. As we noted above in the introduction, in Gul and Pesendorfer’s (2014) EUU theory, this requires both it and its complement satisfing Savage’s (1954) postulate **P2**. However, for the reasons we presented in the introduction, we propose the following alternative property of *weak decomposability* as the one an event and its complement need to satisfy in order for us to infer the DM deems that event and its complement as “risky” and hence “measurable” by her prior.

Definition 6 (Decomposable Splits) *An event* $R \subseteq \Omega$ *and its complement* $\Omega\backslash R$ *constitute a* decomposable split *(of the state space* Ω*) if for every pair of acts* f *and* g *in* F,

$$f_R g \succ g \textit{ and } g_R f \succ g \implies f \succ g\,.$$

Set $\mathbf{R} := \{R \subseteq \Omega \colon \{R\,,\Omega\backslash R\}$ *constitutes a decomposable split* $\}$.

We shall refer to $\mathbf{R}$ as the set of decomposable events. Notice that by construction the set $\mathbf{R}$ (like the set of ideal events) is closed under complements. Also it is immediate from their respective definitions that if $\succsim$ is complete and transitive then any ideal split of the state space is also a decomposable split.[10] The converse, however, need not hold.

Recall from the introduction an act is *decomposable* if it is measurable with respect to $\mathbf{R}$, that is, an act g is decomposable if $g^{-1}(\{x\}) \in \mathbf{R}$ for all $x \in X$. Let $G \subset F$ denote the set of decomposable acts.

A subclass of decomposable events are those for which modifying any act on that event leaves it in the same indifference set. These are known as (Savage-)null events.

Definition 7 (Null events) *An event $N \subseteq \Omega$ is* null *if $f \sim g_N f$ for all $f, g \in F$. Let $\mathbf{N}$ denote the set of null events.*

Set $\mathbf{R}^+ := \mathbf{R} \backslash \mathbf{N}$, the class of non-null decomposable events. And for each non-null decomposable event $R \in \mathbf{R}^+$ and each act $f \in F$, set $f(R)^+ := \{y \in f(R) \colon f^{-1}(y) \cap R \notin \mathbf{N}\}$. That is, $f(R)^+$ contains each element in the image of $f(R)$ whose *inverse image* has a non-null intersection with R.

Analogous to the role played by ideal events in Gul and Pesendorfer (2014), we suppose an ILUU maximizer uses elements of $\mathbf{R}^+$ to quantify the uncertainty of any event. So it seems natural to view an event as *maximally ambiguous* if it and its complement contain no element of $\mathbf{R}^+$. Adopting the terminology of Gul and Pesendorfer (2014), we will refer to such an event (as well as its complement) as *diffuse*.

Definition 8 (Diffuse Events) *An event $D \subseteq \Omega$ is* diffuse *if, for every non-null decomposable event $R \in \mathbf{R}^+$, both $D \cap R \notin \mathbf{N}$ (that is, D has a non-null intersection with R) and $(\Omega \backslash D) \cap R \notin \mathbf{N}$ (that is, the complement of D has a non-null intersection with R). We denote by $\mathbf{D}$ the set of diffuse events.*

We say an act h is *diffuse* if its inverse image generates a diffuse partition of Ω, by which we mean $h^{-1}(x) \in \mathbf{D}$ for all $x \in h(\Omega)^+$. Let $H \subset F$ denote the set of diffuse acts.

Finally, we say a sequence of acts $\{f^n\}_{n=1}^{\infty}$ in F *converges in preference* to f if for any two acts f_* and f^* in F satisfying $f^* \succ f \succ f_*$ there exists an integer N such that $f^* \succ f^n \succ f_*$ whenever $n \geqslant N$.

With these preliminaries in hand we can now state the axioms. Our first is the standard ordering axiom.

Axiom 1 (Ordering) *The binary relation $\succsim$ is complete and transitive.*

[10] If the split $\{E, \Omega \backslash E\}$ is ideal, then $g_E f \succsim f$ implies $g \succsim f_E g$ or equivalently, $\neg(g \succsim f_E g)$ implies $\neg(g_E f \succsim f)$. It follows from the completeness of $\succsim$ that $f_E g \succ g$ implies $f \succ g_E f$. Hence if, in addition, we have $g_E f \succ g$, then $f \succ g$ follows from the transitivity of $\succ$, which in turn follows from the completeness and transitivity of $\succsim$.

We next require the collection of events the DM deems unambiguous to be closed under conjunctions. That is, we require for any pair of decomposable splits $\{R, \Omega\backslash R\}$ and $\{\widehat{R}, \Omega\backslash\widehat{R}\}$, the two-element partition $\{R\cap\widehat{R}, \Omega\backslash R\cup\Omega\backslash\widehat{R}\}$ is also a decomposable split.

Axiom 2 (Decomposable Conjunction Closure) *For any pair of decomposable events $R, \widehat{R} \in \mathbf{R}^+$ and any pair of acts $f, f' \in \mathcal{F}$,*

$$\text{if } \; f_{R\cap\widehat{R}}f' \succ f' \;\text{ and }\; f'_{R\cap\widehat{R}}f \succ f' \;\text{ then } f \succ f' .$$

We readily acknowledge this is not without loss of generality. As a number of researchers have demonstrated, various hedging opportunities against ambiguity may be based on events one would expect a DM to deem unambiguous but which are not closed under intersection.[11] However, just as was the case in Gul and Pesendorfer's (2014) model of EUU, as our theory builds on the induced order over the (meaurable) envelopes the DM associates with acts, we require the set of events the DM deems measurable, constitute a σ-algebra thereby guaranteeing the (measurable) envelope the DM associates with each act is well-defined. Axiom 2 in conjunction with the continuity property in part 2 of Axiom 5 (see below), plays a key role in establishing that the collection of events $\mathbf{R}$ is a σ-algebra.

Recall from above, a diffuse event is one in which the DM cannot find any non-null decomposable event that can "fit" inside that event that would help her quantify the uncertainty she associates with that event. Correspondingly, for a given diffuse act $h \in H$, the DM is unable to estimate the *relative* likelihood of *any strict subset* of the objects in $h(\Omega)^+$ obtaining either unconditionally or conditionally on any decomposable event R obtaining. Thus we can view the restriction of the her preferences to $H \cup X$ (respectively, to $\{h_R f : h \in H \cup X\}$ for some fixed decomposable event $R \in \mathbf{R}$ and some fixed act $f \in F$) as revealing her (conditional) preference between pairs of sets of objects. The next axiom requires these derived (conditional) preferences over outcome-sets exhibit the appropriate monotonicity (c.f. Definition 4). In this respect we view the axiom as a natural extension of Savage's **P3** to outcome-sets.

Axiom 3 (Outcome-Set Monotonicity) *Fix a pair of acts h and $\widehat{h}$ in $H \cup X$. If $h(\Omega)^+$ dominates $\widehat{h}(\Omega)^+$, then $h_R f \succ \widehat{h}_R f$ for all $f \in F$ and $R \in \mathbf{R}^+$.*

We follow with an axiom that serves the role played by Savage's comparative probability axiom **P4** in Gul and Pesendorfer's (2014) characterization of EUU. However, since the decomposability property does not entail the full separability implied by the definition of an ideal event, it is not enough for us to simplify modify **P4** by substituting decomposable events for ideal events. So instead we adapt the conditional weak comparabtive probability axiom $\mathbf{P4}^c$ from Epstein and LeBreton (1993) with an extension to diffuse acts.

[11] See for example, Zhang (2001), Epstein and Zhang (2001), Kopylov (2007), and Nehring (2009)

Axiom 4 (Conditional Equal Probability) *For any three decomposable events $R, \widehat{R}, T \in \mathbf{R}$, $R \cup \widehat{R} \subseteq T$, any pair of outcomes $x^* \neq x$ in X, and any act $f \in F$:*

$$(x^*_R x)_T f \sim (x^*_{\widehat{R}} x)_T f \implies (h^*_R h)_T f \sim (h^*_{\widehat{R}} h)_T f\,,$$

for all h^* and h in $H \cup X$.

To interpret this axiom, notice that the first indifference allows us to infer that, conditional on the act f determining the outcome should the event T *not* obtain, the DM views the event R given T is "as equally as likely" as the event $\widehat{R}$ given T. The axiom then requires these revealed conditional equal likelihoods should still obtain no matter what pair of outcome-sets serve as the "stakes" for the (conditional on T) bets based on R and $\widehat{R}$, respectively.

We finish with two continuity axioms. As the first deals with continuity with respect to outcome sets we call it *Outcome-Set Continuity.*

Axiom 5 (Outcome-Set Continuity)

1. *If f^n converges uniformly to f with $|f(R')^+| = |f^n(R')^+|$ for all $R' \in \mathbf{R}$ contained in $R \in \mathbf{R}$ then $f^n_R f$ converges in preference to f.*

2. *If $f^n = f_{R^n} f''$ with $\{R^n\} \subset \mathbf{R}$ and $R^n \subset R^{n+1}$ then f^n converges in preference to $f_{\cup_n R^n} f''$.*

This form of continuity may seem novel but we motivate the first part by noting it allows for the type of discontinuities that arise in the representation of QAMUU maximizers as illustrated in Example 1. The second part helps ensure the collection of events the DM deems unambiguous is closed under countable unions.

Our second continuity axiom which were refer to as *Small Event Continuity*, plays the same role as Savage's (1954) postulate **P6**, namely, requiring the set of decomposable events is sufficiently rich so that the derived comparative likelihood relation over decomposable events is both fine and tight. This in turn ensures there exists an agreeing probability for this derived relation.

Axiom 6 (Small Event Continuity) *For any pair of acts $f, f' \in \mathcal{F}$, if $f \succ f'$, then for each outcome $x \in X$ there exists a finite decomposable partition $\{R_1, \ldots, R_N\}$ of Ω, with $R_n \in \mathbf{R}$, such that $x_{R_n} f \succ f'$ and $f \succ x_{R_n} f'$ for all $n = 1, \ldots, N$.*

Our main representation result follows.

Theorem 1 *Fix a binary relation $\succsim$. Then the following are equivalent.*

1. *The relation $\succsim$ satisfies Axioms 1 – 6.*

2. *The relation $\succsim$ is a member of the class of ILUU maximizers.*

We conclude this section by considering an additional monotonicity axiom, that formed part of Gul and Pesendorfer's (2014) characterization of EUU maximization.

Axiom 7 (Statewise Monotonicity) *For any pair of acts* $f, \widehat{f} \in F$, $f \succ \widehat{f}$ *whenever* $f \gg \widehat{f}$.

Although this monotonicity property may seem natural in the present setting in which outcomes can be viewed as monetary prizes, adding it to the other six axioms reduces outcome-set utilities to interval utilities.

Proposition 2 *Fix a prior* μ *and balanced outcome-set utility* U. *Then the following are equivalent.*

1. *The preference relation* $\succsim$, *characterized by* $\langle \mu, U \rangle$, *satisfies Axiom 7.*
2. *For all* $Y \in \mathcal{X}$ *and all* $c \in X^0$, $U(Y, c) = U(\{\min_{y \in Y}, \max_{y \in Y}\}, c)$.

To get a flavor as to why this should be the case, consider a diffuse act h with outcome set Y with minimum y and maximum x. By replacing all the intermediate outcomes in Y by x (respectively, y), we obtain a diffuse act h' (respectively, h''). Since both h' and h'' have the same outcome set $\{x, y\}$ they must have the same value. Furthermore, since $h' \geqslant h \geqslant h''$, this in turn means that h must also have the same value when Axiom 7 holds true. By considering the ILUU representation of acts of the form $h_R y \sim c$, it then follows that $U(Y, c) = U(\{x, y\}, c)$ for all $c \in X^0$. Conversely, by comparing the interval-envelopes of acts $f \gg \widehat{f}$, it readily follows that the subclass of ILUU with interval utilities satisfies statewise monotonicity.

In order to accommodate outcome-set utilities that cannot be reduced to interval utilities, Axiom 7 clearly cannot be part of the characterization of ILUU maximization. We contend, however, its omission accords with the (implicit) assumption we maintain throughout that the DM's knowledge about those aspects of an act she deems relevant, is necessarily restricted to its associated envelopes.

5 Consistent Planning Optimality

As is well known, to extend to a dynamic setting any preference model in which a DM exhibits non-neutral attitudes toward the ambiguity she perceives to be present generally entails the violation of at least one of two principles that are typically considered normatively desirable: *consequentialism*, the requirement that a DM's conditional preferences depend only on outcomes on states that are still possible; and *dynamic consistency*, the requirement that any consistent plan (that is, one that the DM will stick to as it maximizes her conditional preferences) is ex ante optimal.

As we outlined in the introduction, if every event the DM deems measurable (and hence lies in the domain of her prior) is ideal, then whenever the partition of conditioning events in a dynamic choice problem are all ideal, her conditional preferences satisfy consequentialism and her choices guided by those conditional preferences generate plans that are ex ante optimal.

In this section, we consider conditional preferences that need not satisfy consequentialism and investigate to what extent it is possible to retain the ex ante optimality of consistent plans with the proviso that any event on which the DM conditions her preferences is required to be "measurable" with respect to the prior that represents her beliefs. More precisely, we consider a DM who exhibits rich coherent beliefs, where the prior μ with domain $\mathbf{R}$ is a coherent belief for the DM's (static) preferences $\succsim$. Furthermore, although she need not be an ILUU maximizer and so, for example, the events in $\mathbf{R}$ need not be decomposable, we assume her preferences $\succsim$ admit a certainty equivalent representation $W\colon \Delta(\mathcal{X}) \to X$, in which for any pair of acts f and g with associated respective envelopes $\mathbf{f}$ and $\mathbf{g}$,

$$f \succsim g \text{ if and only if } W(\mu \circ \mathbf{f}^{-1}) \geqslant W(\mu \circ \mathbf{g}^{-1})\,.$$

We require W satisfies the following monotonicity and continuity properties.

(i) Outcome-Set Monotonicity: for any $L \in \Delta(\mathcal{X})$, and any Y and Z in $\mathcal{X}$:
Y dominates Z implies $W(\alpha\delta_Y + (1-\alpha)L) > W(\alpha\delta_Z + (1-\alpha)L)\,$, for all $\alpha \in (0\,,1]$.

(ii) Mixture Continuity: for any three outcome-set lotteries L, L' and L'' in $\Delta(\mathcal{X})$,
if $W(L) > W(L') > W(L'')$, then the sets $\{\alpha \in [0,1]\colon W(\alpha L + (1-\alpha)L'') > W(L')\}$ and $\{\alpha \in [0,1]\colon W(\alpha L + (1-\alpha)L'') < W(L')\}$ are open.

Below we consider the DM's behavior in the following class of dynamic decision problems.

Definition 9 (Information Decision Problems) *An* information decision problem (IDP) *d is characterized by a signal and action-set pair $\langle (S\,,\sigma)\,,A\rangle$, in which S is a (finite) signal space corresponding to the partition of the state space Ω described by the (onto) function $\sigma\colon \Omega \to S$ and $A \in \mathcal{F}$ is the (finite) set of available acts from which the DM selects after learning the signal's realization.*

In the IDP $d = \langle (S\,,\sigma)\,,A\rangle$, the DM learns the signal realization is s when the event $\sigma^{-1}(s)$ obtains. However, given her partial understanding of the data generating process, the most she can infer from receipt of s is that $\overline{\sigma^{-1}(s)}$, the outer-sleeve of $\sigma^{-1}(s)$, has obtained. The complication that this raises is that the collection of measurable events $\{\overline{\sigma^{-1}(s)}\}_{s\in S}$ is not a partition of the state space unless σ is itself measurable with respect to $\mathbf{R}$.[12] However, we will suppose that the DM treats this collection as though it were a partition and that she associates with it an appropriately defined *conditional pseudo-belief system* that encodes how she updates her beliefs given the signal.

To illustrate and motivate the approach we take, consider first the case of a binary signal σ with signal space $S = \{1\,,2\}$. Associated with the signal σ is its measurable partition $\{R_\sigma^{\{1\}}\,,R_\sigma^{\{2\}}\,,R_\sigma^{\{1,2\}}\}$.

[12] If $\sigma^{-1}(s) \in \mathbf{R}$ then $\overline{\sigma^{-1}(s)} = \underline{\sigma^{-1}(s)} = \sigma^{-1}(s)$

When observing the signal realization s, the DM just knows that its outer-sleeve, $R^s_\sigma := R^{\{s\}}_\sigma \cup R^{\{1,2\}}_\sigma$ has obtained. Thus the DM considers the measurable event $R^{\{1,2\}}_\sigma$ (the σ-marginal inner-sleeve of $\{1\,,2\}$) is still possible after receipt of either of the signal's two possible realizations. To avoid any "double-counting" and in light of her lack of knowledge about which of the two intersections $\sigma^{-1}(1) \cap R^{\{1,2\}}_\sigma$ and $\sigma^{-1}(2) \cap R^{\{1,2\}}_\sigma$ is ex ante more likely, assume the DM employs a form of "principle of insufficient reason" and so attributes one-half of the probability $\mu\left(R^{\{1,2\}}_\sigma\right)$ to each of these two possible signal realizations. So, for each measurable event R in $\mathbf{R}$ she sets

$$\mu_\sigma\,(R\,,s) := \mu\left(R \cap R^{\{s\}}_\sigma\right) + \frac{\mu\left(R \cap R^{\{1,2\}}_\sigma\right)}{2}\,,$$

We interpret $\mu_\sigma\,(R\,,s)$ as the *pseudo-prior* belief the DM assigns to the signal realization being s and the measurable event R obtaining. It then follows for every R in $\mathbf{R}$:

$$\mu_\sigma\,(R\,,1) + \mu_\sigma\,(R\,,2) = \mu(R)\,.$$

That is, the pseudo-prior preserves the total prior probability that her actual prior belief μ assigns to each measurable event. Furthermore, after observing the signal realization s, we can define the DM's *pseudo-posterior* belief $\mu_\sigma(\cdot\,|\,s)$ by setting for each measurable event R in $\mathbf{R}$

$$\mu_\sigma(R\,|\,s) := \frac{\mu_\sigma\,(R\,,s)}{\mu_\sigma\,(\Omega\,,s)}\,.$$

More generally, consider $\{R^Q_\sigma \in \mathbf{R}\colon Q \in \mathcal{S}\}$, the measurable partition associated with the signal $(\sigma\,,S)$. It follows from definition 2 that for each $Q \in \mathcal{S}$:

$$\bigcup_{Q\in\mathcal{S}\,:\;s\in Q} R^Q_\sigma \supseteq \sigma^{-1}(Q) \text{ and } \mu\left(\bigcup_{Q\in\mathcal{S}\,:\;s\in Q} R^Q_\sigma\right) = \sum_{Q\in\mathcal{S}\,:\;s\in Q} \mu\big(R^Q_\sigma\big) = \mu\left(\overline{\sigma^{-1}\,(s)}\right).$$

So for the signal $(S\,,\sigma)$ we shall assume the DM interprets the receipt of the signal realization $s \in S$ as evidence that the event

$$R^s_\sigma := \bigcup_{Q\in\mathcal{S}\,:\;s\in Q} R^Q_\sigma \left(=\overline{\sigma^{-1}(s)}\right)\,,$$

has obtained and she will associate with this signal a conditional pseudo-belief system. A formal definition follows.

Definition 10 (Conditional pseudo-belief system) *Fix a signal $(S\,,\sigma)$ with associated measurable partition $\{R^Q_\sigma \in \mathbf{R}\colon Q \in \mathcal{S}\}$. A conditional pseudo-belief system associated with this signal comprises a collection of pairs of pseudo-prior and pseudo-posterior beliefs $\left(\mu_\sigma\,(\cdot, s)\,, \mu_\sigma(\cdot\,|\,s)\right)_{s\in S}$, where for each $s \in S$, $\mu_\sigma\,(\cdot, s) : \mathbf{R} \to [0\,,1]$ is an additive non-negative measure. Furthermore, for each $R \in \mathbf{R}$:*

1. *for each* $s \in S$,

$$\mu_\sigma(R \mid s) = \frac{\mu_\sigma(R\,,s)}{\mu_\sigma(\Omega\,,s)};$$

2. *for each* $Q \in \mathcal{S}$*, each* $s \in Q$*, and any* $R' \in \mathbf{R}$ *such that* $R' \subseteq R_\sigma^Q$,

$$\left(\mu_\sigma(R \cap R_\sigma^Q \mid s) - \mu_\sigma(R' \mid s)\right)\left(\mu(R \cap R_\sigma^Q) - \mu(R')\right) > 0\,,$$

whenever $\mu(R \cap R_\sigma^Q) \neq \mu(R')$;

3. $\sum_{s\in S} \mu_\sigma\left(R\,,\,s\right) = \mu\left(R\right)$.

Condition 1 is simply the law of conditional probabilities applied to each pseudo-prior and -posterior belief pair. Condition 2 may be viewed as a consistency requirement between the prior and the pseudo-posteriors. Specifically, it requires that the relative ranking of the prior probability of any measurable event's intersection with an element of the measurable partition associated with σ and another measurable subset of that element of the measurable partition is preserved by the pseudo-posterior. Finally, condition 3 may be viewed as a form of the law of total probability in that it requires the sum of the probabilities assigned to a measurable event by the pseudo-prior beliefs must equal the probability assigned to that event by the prior.

The following lemma provides a simple characterization of the class of pseudo-belief systems that can be associated with a signal.

Proposition 3 *Fix a signal* (S, σ) *and an associated conditional pseudo-belief system* $\left(\mu_\sigma(\cdot, s)\,, \mu_\sigma(\cdot \mid s)\right)_{s\in S}$*. There exist vectors of (non-negative) weights* $\{\lambda^Q \in \Delta(Q) \colon Q \in \mathcal{S}\}$ *such that*

$$\mu_\sigma\left(R, s\right) = \sum_{Q\in\mathcal{S}\colon\, s\in Q} \lambda_s^Q\ \mu\left(R \cap R_\sigma^Q\right) \ \text{for each } R \in \mathbf{R} \text{ and each } s \in S\,.$$

One straightforward way to generate a pseudo-belief system is to assign equal weights to the elements in each Q. That is, for each $Q \in \mathcal{S}$ and each $s \in Q$ take λ_s^Q to be $|Q|^{-1}$, so that for each $R \in \mathbf{R}$ we set

$$\mu_\sigma\left(R \cap R_\sigma^Q\,, s\right) := \frac{\mu(R \cap R_\sigma^Q)}{|Q|}\,.$$

Notice, if we consider the cooperative game in which the set of "players" are taken to be S and the value of each coalition $Q \in \mathcal{S}$ of "players" is given by $v(Q) := \mu\left(\underline{\sigma^{-1}(Q)}\right)$, then $\mu(R_\sigma^Q)$, the probability the prior assigns to the ϕ-marginal inner-sleeve of Q, corresponds to the *Harsanyi dividend* generated by the "coalition" Q. Hence the use of equal weights can be viewed as employing the equal Harsanyi dividend allocation rule in sharing this surplus among the "players" in a "coalition". This in turn, implies the pseudo-prior probability $\mu_\sigma(\Omega\,, s)$ corresponds to the *Shapley value* for the "player" s. (See Harsanyi (1982).)[13]

[13] An alternative approach taken by Gul and Pesendorfer (2021) is to assume the DM interprets the receipt of the

Within each given decision problem $d = \langle (S\,,\sigma)\,,A\rangle \in D$, the DM chooses an act $f \in A$ for each signal realization $s \in S$. Take $S = \{1,\ldots,n\}$, and let $b = (f_1,\ldots,f_n)$ denote a generic *behavior* for d. After the receipt of the signal realization s, the DM associates with the act f the outcome-set lottery $\mu_\sigma\left(\mathbf{f}^{-1}(\cdot) \mid s\right) \in \Delta(\mathcal{X})$. Hence ex ante, the DM associates with the behavior $b = (f_1,\ldots,f_n)$, the outcome-set lottery

$$\sum_{s\in S} \mu_\sigma\,(\Omega\,,s\,)\;\mu_\sigma\left(\mathbf{f}_s^{-1}(\cdot) \mid s\right)\,.$$

Notice that it follows from condition 3 of Definition 10 that any constant behavior $(f,\ldots,f)$ is associated ex ante with the outcome-set lottery $\mu \circ \mathbf{f}^{-1}$.

The ex ante optimal behavior (or *plan of action*) of the DM in the decision problem $d = \langle (S,\sigma), A\rangle$ can thus be characterized as a solution to

$$\max_{(f_1,\ldots f_n)\in A^n} W\left(\sum_{s\in S} \mu_\sigma\,(\Omega\,,s\,)\;\mu_\sigma\left(\mathbf{f_s}^{-1}(\cdot) \mid s\right)\right)\,. \tag{8}$$

How the DM actually behaves in a decision problem will be assumed to be guided by her (potentially non-consequentialist) *conditional* preferences in which the dependence need not be restricted to which element of the pseudo-partition associated with the signal she is informed has obtained. Rather, it may also depend on what was her ex-ante plan of action. That is, given the ex ante plan of action $\bar{b} = \left(\bar{f}_1,\ldots,\bar{f}_n\right)$, the DM's preferences over the acts f in A that will guide her *actual* choice conditional on the receipt of the realization s are represented by the functional:

$$W\left(\mu_\sigma\,(\Omega\,,s\,)\;\mu_\sigma\left(\mathbf{f}_s^{-1}(\cdot) \mid s\right) + \sum_{s'\neq s} \mu_\sigma\,(\Omega\,,s'\,)\;\mu_\sigma\left(\bar{\mathbf{f}}_{s'}^{-1}(\cdot) \mid s'\right)\right) \tag{9}$$

We assume the DM's actual behavior in a decision problem is a consistent plan, that is, one that she will stick to as it maximizes her conditional preferences. The formal definition follows.

Definition 11 (Consistent Plan) *Fix a decision problem $d = \langle (S\,,\sigma)\,,A\rangle$. The behavior $\widehat{b} = \left(\widehat{f}_1,\ldots,\widehat{f}_n\right)$ is a* consistent plan *if*

---

signal realization $s \in S$ *as though* she has learnt that the inner-sleeve of $\sigma^{-1}(s)$, that is, the event $\underline{\sigma^{-1}(s)}\ \left(= R_\sigma^{\{s\}}\right)$, has obtained. That is, it is as though the DM replaces her prior μ with the "proxy" (or pseudo-prior) $\mu_\sigma(R\,,s)$ given by

$$\mu_\sigma(R\,,s) := \frac{\mu\left(R \cap \underline{\sigma^{-1}(s)}\right)}{\sum_{s'\in S}\mu\left(R \cap \underline{\sigma^{-1}(s')}\right)}\,.$$

Although Gul and Pesendorfer's (2021) proxy does not satisfy the third condition in Definition 10, it does satisfy the first condition as well as the second condition restricted to $Q \in \mathcal{S}$ for which $|Q| = 1$. Moreover, Theorem 4 below also holds for this alternative way the DM interprets a signal's realization, since for each possible realization of the signal the DM still conditions on a measurable event.

$$
\begin{aligned}
&W\left(\mu_\sigma(\Omega, s)\, \mu_\sigma\left(\widehat{\mathbf{f}}_s^{-1}(\cdot) \mid s\right) + \sum_{s' \neq s} \mu_\sigma(\Omega, s')\, \mu_\sigma\left(\widehat{\mathbf{f}}_{s'}^{-1}(\cdot) \mid s'\right)\right) \\
&\geqslant W\left(\mu_\sigma(\Omega, s)\, \mu_\sigma\left(\mathbf{f}^{-1}(\cdot) \mid s\right) + \sum_{s' \neq s} \mu_\sigma(\Omega, s')\, \mu_\sigma\left(\widehat{\mathbf{f}}_{s'}^{-1}(\cdot) \mid s'\right)\right), \qquad (10)
\end{aligned}
$$

for all $f \in A$ and all $s \in S$.

Viewing each signal realization as a "player" in a normal-form game with common strategy set A and payoff function given in expression (9), a consistent plan can be interpreted as a Nash equilibrium of the agent-normal form of the DM's dynamic decision problem. Existence of a behavior $\widehat{b}$ that satisfies inequalities (10) follows from the standard existence result for (finite) *Potential* games.[14]

Remark 6 *Notice that if outcome-set monotonicity is strengthened to (mixture) independence, that is, we require for all outcome-set lotteries $L', L'', L \in \Delta(\mathcal{X})$, and all $\alpha \in (0,1)$, $W(L') > W(L'') \implies W(\alpha L' + (1-\alpha)L) > W(\alpha L'' + (1-\alpha)L)$; then for each R in $\mathbf{R}$, the two-element partition $\{R, \Omega \backslash R\}$ is an ideal split of the state space. Moreover, the conditional preferences discussed above become consequentialist since inequality* (10) *would now entail*

$$
W\left(\mu_\sigma\left(\widehat{\mathbf{f}}_s^{-1}(\cdot) \mid s\right)\right) \geqslant W\left(\mu_\sigma\left(\mathbf{f}^{-1}(\cdot) \mid s\right)\right),
$$

for all $f \in A$ and all $s \in S$.

Informally, it is natural to view a DM as dynamically consistent if her behavior within each decision problem induces an outcome-set lottery that is best among those outcome-set lotteries that were achievable in that decision problem. The following result establishes that a necessary and sufficient condition for this to hold is for every event in the domain of the DM's prior to be decomposable.

Theorem 4 *Suppose a DM's static preferences $\succsim$ admit a certainty equilivalent representation $W\left(\mu\left(\mathbf{f}^{-1}(\cdot)\right)\right)$ where μ (with domain $\mathbf{R}$) is a countably-additive and convex-ranged probability, $\mathbf{f}$ is the envelope of f and $W\colon \Delta(\mathcal{X}) \to X$ is a function that satisfies outcome-set monotonicity and mixture continuity.*

Further suppose, the DM associates with each signal (S, σ) a conditional pseudo-belief system $\left(\mu_\sigma(\cdot, s), \mu_\sigma(\cdot \mid s)\right)_{s \in S}$. Then the following are equivalent.

1. *For each decision problem, every consistent plan is ex ante optimal.*
2. *Each event in the domain of μ and its complement constitutes a decomposable split.*

[14]See Monderer and Shapley (1996). As this is a game of "common interest", the potential function is simply the payoff function.

Given the hypothesis of Theorem 4 for properties assumed to hold for $\succsim$, we have as an immediate corollary that every consistent plan is ex ante optimal if and only if $\succsim$ is an ILUU maximizer.

Sketch of Proof of Theorem 4

A formal proof appears in the appendix, but we provide here an outline of the argument.

1. implies 2.

We establish this holds by proving the contrapositive, namely, if there exists a measurable event $\widehat{R} \in \mathbf{R}$ that is not decomposable, then we can find a decision problem for which there exists a consistent plan that is not ex ante optimal.

A decision problem where such a failure of dynamic consistency occurs, is particularly straightforward to specify once we establish the following: given the monotonicity and continuity properties of $\succsim$ that are implied by the representation, if $\widehat{R} \in \mathbf{R}$ is not decomposable then there exists a pair of acts f and g for which $f \succ g_{\widehat{R}}f$, $f \succ f_{\widehat{R}}g$ *and yet* $g \succ f$.

So consider the (unambiguous signal) decision problem d with $S = \{1, 2\}$, $\sigma(\omega) = 1$ if $\omega \in \widehat{R}$ and $\sigma(\omega) = 2$ if $\omega \notin \widehat{R}$, and $A = \{f, g\}$. The behavior $\widehat{b}(1) = \widehat{b}(2) = f$ is a consistent plan but the act corresponding to the behavior $b^*(1) = b^*(2) = g$ is ex ante superior.

2. implies 1.

We first show this holds for all unambiguous-signal decision problems, by which we mean problems for which the event $\sigma^{-1}(s)$ is decomposable for all s. To establish any consistent plan $\widehat{b}$ for the unambiguous-signal decision problem d is dynamically consistent, we first prove the following property for decomposable partitions: if there is no strict incentive to move from one act toward another act on any single element of that partition, then there is no strict incentive to move from the former to the latter globally. Or a little more formally, if $\{R_1, \ldots, R_n\}$ is a decomposable partition of the state space and $f \succsim g_{R_i}f$ for all $i = 1, \ldots, n$ then $f \succsim g$.

Since by definition, there is no strict incentive to move away from the act prescribed by $\widehat{b}$ for each event $\sigma^{-1}(s)$, there can be no strict incentive to move away globally by the act generated by $\widehat{b}$ to any other act that can be achieved by any other possible behavior b in the decision problem d. Hence $\widehat{b}$ is ex ante optimal.

To complete the argument we show for any arbitrary decision problem $d = \langle (S, \sigma), A\rangle$ there exists an *unambiguous equivalent* problem $d' = \langle (S, \sigma'), A\rangle$ which only differs by having a signal that induces a decomposable partition of the state space in a way that makes the behaviors of the two problems *identical* in terms of the outcome-set lotteries they induce. That is, the behavior b' in the unambiguous-signal decision problem d' induces the same ex ante and conditional outcome-set lotteries as the behavior $b = b'$ in decision-problem d. Hence as we have already established that $\widehat{b}'$ is ex ante optimal in the unambiguous-signal problem d' it necessarily follows that it is also ex ante optimal in d.

6 Concluding Comments

Like Gul and Pesendorfer's (2014) EUU model, the ILUU model affords the outside observer the ability to infer those events the DM deems measurable, *solely* from her preferences. And similar to Gul and Pesendorfer (2014), these may be viewed as those events for which Savage's (1954) *sure-thing principle* applies. Adapting the quote by Savage from the introduction, informally, we might phrase it in the following way:

> *The event R is deemed to be measurable by the DM, if for any pair of acts f and g, for which she prefers g to f, were she to know that the event R obtains, as well as were she to know that the complement of the event R obtains, she must prefer g to f.*

Unlike Gul and Pesendorfer, however, we have not "operationalized" this principle by imposing Savage's postulate **P2**. Instead, following Grant et al. (2000), we have interpreted statements like *"g would be preferred to f if the event R were known to obtain"* as only entailing $g_R f \succ f$. Hence, our notion of when one may infer the DM deems an event R to be measurable corresponds to the requirement that for every pair of acts f and g,

$$g_R f \succ f \text{ and } g_{\Omega \setminus R} f \ \ (= f_R g) \succ f \text{ implies } g \succ f\,.$$

That is, the event and its complement constitute a *decomposable split* of the state space.

As a consequence and following as an immediate corollary to Theorem 1, for a DM who exhibits rich coherent beliefs, her risk preferences (over outcome lotteries), which can naturally be identified with the restriction of her preferences to measurable acts, need only satisfy the betweennness property of Chew (1983) and Dekel (1986), rather than (full) independence. Moreover, the two ILUU sublcasses, Weighted Uncertain Utility and Disappointment Averse Uncertain Utility outlined in section 3 provide us with a pair of parsimoniously parameterized models that not only can accommodate Ellsberg style choice patterns for decisions involving ambiguous acts but can also produce Allais style choice patterns for risky (that is, measurable) acts.

Preferences that exhibit betweenness properties have been relatively understudied in the context of choice under risk and, to our knowledge, are almost completely absent in the ambiguity literature. However, in light of the above discussion as well as Theorem 4 and the tight connection it draws between the optimality of consistent plans and measurable events that are decomposable, we contend the ILUU family of preferences provides us with a normatively attractive approach for modelling choice under uncertainty when decision makers possess only incomplete information about the stochastic process resolving the uncertainty they face.

Appendix

Proof of Theorem 1

Sufficiency of the axioms

Preliminary Results

We begin with the preliminary result that the set of decomposable events constitutes a σ-algebra: that is, 1) it contains both the universal set Ω and the empty set, $\emptyset$; 2) it is closed under complements; 3) it is closed under intersection; and, 4) it is closed under countable unions.

Lemma 5 *The set of decomposable events* $\mathbf{R}$ *is a* σ*-algebra.*

Proof. 1) From the definition of a decomposable split, it is immediate that $\emptyset \in \mathbf{R}$ and $\Omega \in \mathbf{R}$. 2) If $R \in \mathbf{R}$, then also by definition we have $\Omega \backslash R \in \mathbf{R}$. 3) Axiom 2 ensures $\mathbf{R}$ is closed under intersection. 4) Finally, let $\{R^n\}$ be a set of decomposable events with $R^n \subset R^{n+1}$. Assume by way of contradiction that $\{\bigcup R^n, \Omega \backslash \bigcup R^n\}$ is not a decomposable split. That is, there exist $f, g \in \mathcal{F}$ such that $f_{\bigcup R^n} g \succ g$ and $g_{\bigcup R^n} f \succ g$ but $g \succsim f$.

Axiom 5 implies that $f_{R^n} g$ and $g_{R^n} f$ converge in preference to $f_{\bigcup R^n} g$ and $g_{\bigcup R^n} f$, respectively. Hence there exists an N such that $f_{R^n} g \succ g$ and $g_{R^n} f \succ g$ for all $n \geqslant N$. Since $\{R^n, \Omega \backslash R^n\}$ is a decomposable split, it follows $f \succ g$, a contradiction. Thus, $\{\bigcup R^n, \Omega \backslash \bigcup R^n\}$ is a decomposable split. Therefore $\mathbf{R}$ is a σ-algebra. ■

The next lemma shows that every non-decomposable act can be expressed in terms of diffuse acts and constants acts on its measurable split. Without loss of generality we take $f(\Omega) = f(\Omega)^+$.

We show some properties of decomposable events. Let F^0 denote the acts that maps from Ω to X^0, where $X^0 = (\ell, m)$ is the interior of X.[15]

Lemma 6 *Let* $\succsim$ *be a relation that satisfies Axiom 1 and Axiom 3. For any decomposable event* $E \in \mathbf{R}$ *and any pair of acts* $f, g \in F$:

1. $f \succ f_E g$ *and* $f \succ g_E f$ *implies* $f \succ g$;

and furthermore, if $\succsim$ *also satisfies Axioms 2 and 6 then;*

2. $f \succsim f_E g$ *and* $f \succsim g_E f$ *implies* $f \succsim g$; *and,*

3. $f_E g \succsim g$ *and* $g_E f \succsim g$ *implies* $f \succsim g$.

Proof. Fix $E \in \mathbf{R}$. We show that statement 1 holds by the same techniques in Grant et al. (2000). Assume by way of contradiction that there exist two acts $f, g \in F$, such that $f \succ f_E g$, $f \succ g_E f$ and $g \succsim f$. We consider two cases

[15] Recall that $m := \max_{x \in X}\{x\}$ and $\ell := \min_{y \in X}\{y\}$.

(a) Suppose $f_E g \succsim g_E f$. Then we have

$$g \succsim f \succ f_E g \succsim g_E f.$$

Set $\widehat{f} := f_E g$ and $\widehat{g} := g_E f$. Notice $\widehat{f}_E \widehat{g} = f$ and $\widehat{g}_E \widehat{f} = g$. Thus,

$$\widehat{g}_E \widehat{f} \succsim \widehat{f}_E \widehat{g} \succ \widehat{f} \succsim \widehat{g}.$$

Since E is decomposable, $\widehat{g}_E \widehat{f} \succsim \widehat{f}_E \widehat{g} \succ \widehat{f}$ implies that $\widehat{g} \succ \widehat{f}$, which contradicts $\widehat{f} \succsim \widehat{g}$.

(b) Now suppose, $g_E f \succ f_E g$. Then we have

$$g \succsim f \succ g_E f \succ f_E g.$$

So again, set $\widehat{f} := f_E g$ and $\widehat{g} := g_E f$, and again notice that $\widehat{f}_E \widehat{g} = f$ and $\widehat{g}_E \widehat{f} = g$. Thus,

$$\widehat{g}_E \widehat{f} \succsim \widehat{f}_E \widehat{g} \succ \widehat{g} \succ \widehat{f}.$$

Since E is decomposable, $\widehat{g}_E \widehat{f} \succsim \widehat{f}_E \widehat{g} \succ \widehat{g}$ implies that $\widehat{f} \succ \widehat{g}$, which contradicts $\widehat{g} \succ \widehat{f}$.

Therefore, we have established that statement 1 holds.

We first show that statement 2 holds for any $f \in F$ and any $g \in F^0$. Assume that there are $f \in F, g \in F^0$, $f \succsim f_E g$, $f \succsim g_E f$ and $g \succ f$. Axiom 2 and Axiom 6 together imply that there exists a non-null decomposable event $R \subset E$ such that $\ell_R g \succ f$. Applying the axioms again to the pair $f, \ell_R g$ yields that $\ell_{R \cup R'} g \succ f$ for some non-null decomposable $R' \subset \Omega \backslash E$. Since $g \in F^0$, Axiom 3 combined with Lemma 12 implies $g_E f \succ (\ell_{R \cup R'} g)_E f$ and $f_E g \succ f_E(\ell_{R \cup R'} g)$. Thus, $f \succ (\ell_{R \cup R'} g)_E f$ and $f \succ f_E(\ell_{R \cup R'} g)$, from which we have the contradiction.

Now consider an act $g \in F$ with $g^{-1}(\ell) = B \notin \mathbf{N}$ and another act $f \in F$, for which $f \succsim f_E g$, $f \succsim g_E f$ and $g \succ f$. When both $E \backslash B$ and $(\Omega \backslash E) \backslash B$ contain an element of $\mathbf{R}^+$, the same argument as above still applies. It remains to adjust the argument for the case with $E \backslash B$ diffuse (the construction for $\Omega \backslash E$ is similar). Consider a small positive decreasing sequence $\{\varepsilon_n\}$ converging to zero, and let g^n denote the act with all outcomes of g on $E \backslash B$ decreased by ε_n. The extra clause for boundary outcomes in Definition 4 implies that $g^n_E f \prec g_E f$ for all n. The same argument now applies when $\ell_{R \cup R'} g$ above is replaced by $g^n_E(\ell_{R'} g)$, for n sufficiently large. A similar argument establishes that statement 3 holds. ■

Proposition 7 *An event R is decomposable. If $f_R f' \sim f_R f'' \sim c$, then $f^*_R f' \sim c$ implies $f^*_R f'' \sim c$ for all $f, f', f'', f^* \in F$.*

Proof. Using the non-strict criterion in Lemma 6, $f_R f' \succsim f_R f''$ and $f_R f' \succsim f^*_R f'$ implies $f_R f' \succsim f^*_R f''$. Similarly, $f_R f' \precsim f_R f''$ and $f_R f' \precsim f^*_R f'$ implies $f_R f' \precsim f^*_R f''$ ∎

We now show the axioms imply that $\succsim$ admits an ILUU representation.

Implicit Linear Utility for Decomposable Acts

Lemma 5 has established that the set of decomposable events, $\mathbf{R}$ is a σ-algebra. Denote by $\succsim_G$ the preference relation over G obtained by restricting $\succsim$ to G. By definition any event $E = g^{-1}(x)$, for some $g \in G$ and some $x \in X$ satisfies Grant et al.'s (2000) property of (weak) decomposability. So to establish $\succsim_G$ admits an implicit linear utility representation, we shall first establish $\succsim_G$ satisfies all the other conditions assumed in Corollary 4 of Grant et al. (2000), namely Savage's (1954) **P1**, **P3**, and **P6**, and Epstein and LeBreton's (1993) **P4**^{c}.

The relation $\succsim_G$ is a weak order since $\succsim$ is by Axiom 1, so **P1** is satisfied. By Axiom 3, for all $x, y \in X$, $x \geqslant y$ if and only if $x \succsim y$. Furthermore, for all $R \in \mathbf{R}^+$, and all $g \in G$, $x \geqslant y$ if and only if $x_R g \succsim y_R g$, so **P3**, is satisfied. **P6** follows directly from Axiom 6. That **P4**^{c} also holds is the claim of the following lemma.

Lemma 8 (P4^c) *For any three non-null decomposable events $R, \widehat{R}, T \in \mathbf{R}^+$, $R \cup \widehat{R} \subseteq T$, any four outcomes $x^* > x$ and $y^* > y$, and any decomposable act $g \in G$:*

$$(x^*_R x)_T g \succsim (x^*_{\widehat{R}} x)_T g \implies (y^*_R y)_T g \succsim (y^*_{\widehat{R}} y)_T g \,.$$

Proof. Fix $g \in G$, $x, x^* \in X$ with $x^* > x$, and $R, \widehat{R}, T \in \mathbf{R}$, with $R \cup \widehat{R} \subseteq T$. To transform the premise of **P4**^{c} into an indifference by shrinking R is not straightforward.

Lemma 9 *There exists $R^* \in \mathbf{R}^+$, such that $R^* \subseteq R$ and $(x^*_{R^*} x)_T g \sim (x^*_{\widehat{R}} x)_T g$.*

Proof. Define $f := (x^*_R x)_T g$ and $f' := (x^*_{\widehat{R}} x)_T \sim c$, $c \in X$, and consider the non-trivial case $f \succ f'$. By Axiom 6 there exists $R_1 \in \mathbf{R}^+$ such that $(x^*_{R \setminus R_1} x)_T g \succ (x^*_{\widehat{R}} x)_T g$. By repeating the argument, this yields an infinite disjoint series $\{R_n\} \subset \mathbf{R}^+$ of subsets of R that satisfies

$$(x^*_{R \setminus (\bigcup_{n=1}^k R_n)} x)_T g \succ c \text{ for all } k \,. \tag{11}$$

Let A denote the collection of all such series, and define $B := \{\bigcup_{n=1}^\infty R_n : \{R_n\} \in A\}$. Note that $B \subset \mathbf{R}$, and that $(x^*_{R \setminus R'} x)_T g \succsim c$ for each $R' \in B$, by Axiom 5.2. We show that we can take $R^* = R \setminus M$ with M a maximal element of B, if it exists, and invoke Zorn's lemma to establish that otherwise we can take M as the upperbound outside B of a chain in B.

Lemma 10 *(Zorn's Lemma) Let $\mathcal{P}$ be a partially ordered set in which each chain C has an upperbound. Then $\mathcal{P}$ has at least one maximal element.*

This applies to B as a partially ordered set, by set-inclusion. First assume B has a maximal element M. We can exclude that $(x^*_{R\backslash M}x)_T g \succ c$, since otherwise the procedure above would determine $R' \in \mathbf{R}^+$ for which still $(x^*_{(R\backslash M)\backslash R'}x)_T g \succ c$, implying that also $M \cup R' \in B$, as limit of the series $(M \cup R', \emptyset, ...) \in A$, contradicting that M is maximal. So $(x^*_{R\backslash M}x)_T g \sim c$.

Next suppose B has no maximal element. By Zorn's lemma, B must contain a chain C with upperbound $\bigcup_{\tilde{R}\in C} \tilde{R} \notin B$. Now we can take $R^* = R\backslash M$ for M this upperbound, since from the fact that $M \notin B$ we can again exclude that $(x^*_{R\backslash M}x)_T g \succ c$. ■

To conclude the proof of Lemma 8, Axiom 4 yields $(y^*_{R^*}y)_T g \sim (y^*_{\widehat{R}}y)_T g$ for all $y^* > y \in X$. Hence by Axiom 3 we have $(y^*_R y)_T g \succ (y^*_{R^*}y)_T g \sim (y^*_{\widehat{R}}y)_T g$ for all $y^* > y \in X$ as required. ■

So applying Corollary 4 of Grant et al. (2000), we have there exists a finitely additive, strongly-continuous probability measure μ on $\mathbf{R}$, an implicit utility $v : X \times \mathbb{R} \to \mathbb{R}$, that is increasing in the first argument, such that a function $W : G \to \mathbb{R}$ that represents $\succsim_G$ solves

$$\sum_{x\in X} v(x, W(g))\mu \circ g^{-1}(x) = 0 \qquad \text{for every} \quad g \in G.$$

For each decomposable act $g \in G$, set $V(g) := x \in X$, as the certainty equivalent $x \sim g$ that solves $v(x, W(g)) = 0$. So we can define the normalized balanced utility $\widehat{U} \colon X \times X \to \mathbb{R}$ by setting $\widehat{U}(x, c) := v(x, W(c))$, satisfying $\widehat{U}(x\,, x) = 0$. Axioms 3 and 5 imply that $\widehat{U}$ is continuous and (stricly) increasing in its first argument. Also, from standard arguments in Chew (1983) and Dekel (1986), it follows that $\widehat{U}(\cdot, c)$ is unique up to a scalar $\lambda_c > 0$, and that these scalars can be chosen such that $\widehat{U}$ is (strictly) decreasing and continuous in its second argument. Such $\widehat{U}$ is unique up to one scalar $\lambda > 0$. Axiom 5.2 implies that $x_{R^1\cup\cdots\cup R^n}y$ converges in preferences $x_{\bigcup_{n=1}^{\infty} R^n}y \sim c$ for some $x > y, c \in X$ and any disjoint sequence of decomposable events $\{R_n\}$. Then we have

$$\lim_{n\to\infty}\sum_{i=1}^{n} \mu(R_n)U(x, c) + (1 - \lim_{n\to\infty}\sum_{i=1}^{n} \mu(R_n))U(y, c) = 0$$

which implies $\lim_{n\to\infty} \sum_{i=1}^{n} \mu(R_n) = \mu(\bigcup_{n=1}^{\infty} R^n)$, that is, μ is countably additive.

Thus we derive from the axioms that $\succsim_{\mathcal{G}}$ has an ILUU representation $U(\{x\}, c) := \widehat{U}(x, c)$, which is unique up to a scalar $\lambda > 0$. For the sake of proof, take the unique representation with $\widehat{U}(\{m\}, (\ell + m)/2) = 1$.

Implicit Linear Outcome-Set Utility for General Acts

Turning first to diffuse acts, given we have established above that the measure μ in the implicit linear representation for $\succsim_G$ is a countably-additive convex-ranged probability defined on the σ-algebra of decomposable events $\mathbf{R}$. The following result is that any pair of diffuse acts with a common (non-null) image must have come from the same indifference set.

Lemma 11 *For any pair of diffuse acts h and h' in H: $h(\Omega)^+ = h'(\Omega)^+$ implies $h_R f \sim h'_R f$ for all $f \in F$ and $R \in \mathbf{R}$.*

Proof. Let diffuse acts h, h' in H, $f \in F$ and $R \in \mathbf{R}^+$ with $h(\Omega)^+ = h'(\Omega)^+ \subset X^0$. Then there is an positive decreasing sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ and $(h + \varepsilon_n) \in H$. Axiom 3 implies that $(h + \varepsilon_n)_R f \succ h'_R f$ and so Axiom 5 implies that $h_R f \succsim h'_R f$. Symmetrically, we have $h'_R f \succsim h_R f$ and so $h_R f \sim h'_R f$.

For any pair of h, $h' \in H$ with $h(\Omega)^+ = h'(\Omega)^+$, there are two sequences of $\{h_n\}$ and $\{h'_n\}$ with $h_n(\Omega)^+ = h'_n(\Omega)^+ \subset X^0$, and h_n, h'_n converges uniformly to h and h' receptively. Axiom 2 then implies that $h_R f \sim h'_R f$ all $f \in F$ and $R \in \mathbf{R}$. ■

Turning now to general acts, fix an arbitrary act f in F.

Lemma 12 *Fix a non-decomposable act $f \in F$ with measurable split $\{R_f^Y \in \mathbf{R} : Y \in \mathcal{X}^{f(\Omega)}\}$. For each non-null element R_f^Y in the measurable split of f in which $|Y| > 1$ there exists a diffuse act $h^Y \in H$ such that $h^Y_{R_f^Y} f = f$.*

Proof. Given a non-decomposable $f \in F$, choose a non-null R_f^Y with $Y = \{y_1, \cdots, y_n\}$, $n > 1$. There exists a sequence of disjoint events $\{B_n\}$ such that $B_i = f^{-1}(y_i) \cap R_f^Y$ for all i. Lemma A2 of Gul and Pesendorfer (2014) show that there exists a diffuse partition $\{D_1, \cdots, D_n\} \in \mathbf{D}$ of Ω. Now define

$$D_1^* = (D_1 \cap (\Omega \backslash R_f^Y)) \cup B_1,$$

$$\cdots\cdots$$

$$D_n^* = (D_n \cap (\Omega \backslash R_f^Y)) \cup B_n.$$

We next show that $\{D_1^*, \cdots, D_n^*\}$ is a diffuse partition of Ω. Assume by way of contradiciton that D_i^* is not a diffuse event for some i. Then there is $R \in \mathbf{R}^+$ such that $R \in D_i^*$. Since $\mathbf{R}$ is a σ-algebra, $(\Omega \backslash (R_f^Y)) \backslash R \in \mathbf{R}$. Moreover, $(\Omega \backslash (R_f^Y)) \backslash R \in B_i$, which contradicts B_i containing no decomposable event. Thus, D_i^* is a diffuse event. It is easy to check that D_i^*s are all disjoint and their union is Ω. Therefore, $\{D_1^*, \cdots, D_n^*\}$ is also a diffuse partition of Ω. Set $h^Y = (D_1^* : y_1, \cdots, D_n^* : y_n)$. Then $h^Y_{R_f^Y} f = f$. ■

Lemma 13 *For any $R \in \mathbf{R}^+$ there is a unique $x \in X$ such that $x_R f \sim f$.*

Proof. We have $m_R f \succsim f \succsim l_R f$ by Axiom 3 (in fact its weak version). Suppose there would exist $x^* \in (l, m]$ such that $x^*_R f \succ f$ and $f \succ (x^* - \varepsilon)_R f$ for small positive ε. Since $(x^* - \varepsilon)_R f \to x^*_R f$ as ε goes to 0, Axiom 5 implies that $f \succsim x^*_R f$, contradicting $x^*_R f \succ f$. Thus there is $x \in X$ such that $x_R f \sim f$, and by Axiom 3 it is unique. ■

Now the extention of U to non-singleton outcome-sets is obvious: for an outcome set Y and value level c, set $U(Y, c) := \widehat{U}(z, c)$ for $z \in X$ the certainty equivalent of (any) act of the form $h_R y \sim c$

with $h \in H$ such that $h(\Omega)^+ = Y$, $R \in \mathbf{R}^+$ and $y \in X$. Axiom 6 guarantees that for all Y and c such an act exists, taking R sufficiently small, and Lemma 13 guarantees the existence of z (and its uniqueness given h, and R).

Before verifying the desired properties of U, we first have to show that U is well defined. Lemma 11 guarantees that the choice of h with prescribed outcome set Y is irrelevant, so it remains to show that the outcome of $U(Y, c)$ does not change when R is replaced by another $R' \in \mathbf{R}^+$.

In case $\mu(R') = \mu(R)$ this follows from Axiom 4, taking for R, $\widehat{R}$ and T in the axiom respectively $R \setminus R'$, $R' \setminus R$, and their union. More generally, the certainty equivalent z' only depends on $\mu(R')$. By considering a split of R' in k parts with equal probability, it follows from Proposition 7 that z' is also the certainty equivalent of h on each part. It follows that z' in fact must be independent of $\mu(R')$, and hence $z' = z$, so U is well defined.

By the same proposition it follows that $h_R f \sim z_R f$ whenever the acts have value c. From Lemma 12 it now follows that for any $f \sim c$,

$$\sum_{Y \in \mathcal{X}^{f(\Omega)}} U(Y, c)\mu(R_f^Y) = 0.$$

Set monotonicity of U is directly implied by Axiom 3, and set continuity is implied by Axiom 5, so indeed this is an ILUU representation of $\succsim$.

Necessity of axioms

Axiom 1 follows from the fact that for any $f \in F$, there exists a unique $c \in X$ such that (4) holds true for $V(f) = c$. Let $\mathbf{R}$ denote the domain of μ, which is a σ-algebra. To show that the events in $\mathbf{R}$ are decomposable, consider an arbitrary $R \in \mathbf{R}$, and fix a pair of acts $f, g \in F$, with $f \sim c \in X$. If $g_R f \succ f$, then

$$\sum_{Y \in \mathcal{X}^{(\Omega)}} U(Y\,, c)\,\mu\left(\mathbf{g}^{-1}\,(Y) \cap R\right) > \sum_{Y \in \mathcal{X}^{f(\Omega)}} U(Y\,, c)\,\mu\left(\mathbf{f}^{-1}\,(Y) \cap R\right)\,. \tag{12}$$

And $f_R g \succ f$ implies that

$$\sum_{Y \in \mathcal{X}^{g(\Omega)}} U(Y\,, c)\,\mu\left(\mathbf{g}^{-1}\,(Y) \cap \Omega\backslash R\right) > \sum_{Y \in \mathcal{X}^{f(\Omega)}} U(Y\,, c)\,\mu\left(\mathbf{f}^{-1}\,(Y) \cap \Omega\backslash R\right)\,. \tag{13}$$

Adding inequalities 12 and 13, we get

$$\sum_{Y \in \mathcal{X}^{g(\Omega)}} U(Y\,, c)\,\mu\left(\mathbf{g}^{-1}\,(Y)\right) > 0\,.$$

Hence $V(g) > c$, that is $g \succ f$.

The necessity of the rest of the axioms follows straightforwardly from the ILUU representation combined with Remark 2, on the envelope of acts of the form $f_R g$. In particular, the envelope of $h_R f$ has outcome-set $h(\Omega)$ on R, and equals $\mathbf{f}$ outside R. The necessity of Axiom 4 is now obvious. Axioms 3 and 5 follow directly from the corresponding properties of U. Axiom 6 follows from the fact that μ is convex-ranged.

It remains to show that any event outside $\mathbf{R}$ is not decomposable. By Lemma 6, when $\succsim$ satisfies 1 as well as Axiom 3, an event E is also decomposable if for any pair of acts $f, f' \in F^0$, $f \succsim f'$ whenever $f \succsim f_E f'$ and $f \succsim f'_E f$ and $f_E f' \succsim f'$ and $f'_E f \succsim f'$. Consider an event $E \subset \Omega$ but $E \notin \mathbf{R}_\mu$. Let $\underline{E}$ (respectively, $\underline{\Omega \backslash E}$) denote the inner-sleeve of E (respectively, $\Omega \backslash E$). By Lemma A2 (p22) of Gul and Pesendorfer (2014), we can partition $E \backslash \underline{E}$ (respectively, $(\Omega \backslash E) \backslash (\underline{\Omega \backslash E})$) into two non-null events B^{11} and B^{12} (respectively, B^{21} and B^{22}). Notice by construction none of the four events B^{11}, B^{12}, B^{21} and B^{22} contain any non-null measurable event.

To show E is not decomposable, consider a pair of outcomes $x, y \in X^0$ with $x > y$. Notice it follows from the monotonicity and set-betweenness properties of the balanced outcome-set utility that $U(\{x\}, y) > U(\{y\}, y) = 0$ and $U(\{x\}, y) \geqslant U(\{x, y\}, y) \geqslant U(\{y\}, y) = 0$. Hence at least one of following two inequalities (i) $U(\{x, y\}, y) > 0$ and (ii) $U(\{x\}, y) > U(\{x, y\}, y)$ must hold.

Consider first, the case $U(\{x, y\}, y) > U(\{y\}, y)$ and the act $f = x_{B^{11} \cup B^{21}} y$. Since $y_E f = x_{B^{21}} y$ and $f_E y = x_{B^{11}} y$ it follows that $\mathbf{f} = \{y\}_E \mathbf{f} = \mathbf{f}_E \{y\}$, (that is, the envelope of each of those three acts are all the same). Since by construction, the measure μ is a coherent belief for the preferences generated by the (implicitly defined) ILUU functional, this means $f \sim y_E f \sim f_E y$. However, since

$$\begin{aligned}
&\left[1 - \mu(\underline{E}) - \mu(\underline{\Omega \backslash E})\right] u(\{x, y\}, y) + \left[\mu(\underline{E}) + \mu(\underline{\Omega \backslash E})\right] u(\{y\}, y) \\
> &\left[1 - \mu(\underline{E}) - \mu(\underline{\Omega \backslash E})\right] u(\{y\}, y) + \left[\mu(\underline{E}) + \mu(\underline{\Omega \backslash E})\right] u(\{y\}, y) \, (= 0\,),
\end{aligned}$$

it follows that $f \succ y$. That is, we have established $y_E f \succsim f$, $f_E y \succsim f$ and yet $f \succ y$. A violation of decomposability.

So now consider the case $U(\{x\}, y) > U(\{x, y\}, y)$ and the pair of acts $f = x_{B^{11} \cup B^{21}} y$ and $f' = y_{\underline{E} \cup (\underline{\Omega \backslash E})} x$. Since $f_E f' = y_{(\underline{\Omega \backslash A}) \cup B^{22}} x$ and $f'_E f = y_{\underline{E} \cup B^{12}} x$ it follows that $\mathbf{f} = \mathbf{f_E f'} = \mathbf{f'_E f}$, that is, all three acts come from the same indifference set and hence $f \succsim f_E f'$ and $f \succsim f'_E f$. However, since

$$\begin{aligned}
&\left[1 - \mu(\underline{E}) - \mu(\underline{\Omega \backslash E})\right] u(\{x\}, y) + \left[\mu(\underline{E}) + \mu(\underline{\Omega \backslash E})\right] u(\{y\}, y) \\
> &\left[1 - \mu(\underline{E}) - \mu(\underline{\Omega \backslash E})\right] u(\{x, y\}, y) + \left[\mu(\underline{E}) + \mu(\underline{\Omega \backslash E})\right] u(\{y\}, y) \, (= 0\,),
\end{aligned}$$

it follows $f' \succ f$, again a violation of decomposability. ∎

Proof of Proposition 2.

Let $\succsim$ be characterized by $\langle \mu, U \rangle$, and satisfy Axiom 7. Given $h, h' \in H$ with $h(\Omega)^+, h'(\Omega)^+ \subset X^0$, $h \geqslant h'$. Let $\widehat{h}_n = h + \varepsilon_n \in H$ with $\varepsilon_n > 0$ and $\lim_{n \to \infty} \varepsilon_n = 0$. Axiom7 implies $\widehat{h}_n \succ h$ for all n. Since $\widehat{h}_n$ converges to h uniformly with $|\widehat{h}_n(\Omega) = h(\Omega)|$ for all n, Axiom 5.1 implies $\widehat{h}_n$ converges to h in preference, that is $h \succsim h'$.

Choose $h = x_D y \in H$ with $D \in \mathbf{D}$ and $x > y \in X^0$. Lemma A2 of Gul and Pesendorfer (2014) implies there are disjoint $D_1, D_2 \in \mathbf{D}$ with $D_1 \cup D_2 = D$. Similarly, there are disjoint $D_1', D_2' \in \mathbf{D}$ with $D_1' \cup D_2' = D^c$. For any z with $x > z > y$, define $h' = x_{D_1} z_{D_2} y$ and $h'' = x_D z_{D_1'} y$. We have $h'' \geqslant h \geqslant h'$ and so $h'' \succsim h \succsim h'$. Since $h'(\Omega) = h''(\Omega)$, $h'' \sim h'$, that is, $h'' \sim h \sim h'$. The same argument gives that for all $h, h' \in H$, $h \sim h'$ when $\max h(\Omega) = \max h(\Omega)$, $\min h(\Omega) = \min h(\Omega)$ and $h(\Omega), h'(\Omega) \subset X^0$.

Let $h, h' \in H$ with $h(\Omega) = \{m, l\}$ and $h'(\Omega) = \{m, x_1, \cdots, x_n, l\}$ with $m > x_1 >, \cdots, > x_n > l$. Given a positive decreasing sequence $\{\varepsilon_n\}$ such that $m - x_1 > \varepsilon_n$, $x_n - l > \varepsilon_n$ and for all n and ε_n goes to 0 as $n \to \infty$. Define h_n, h_n' as follows:

$$\begin{aligned} &h_n(\omega) = h(\omega) - \varepsilon_n \text{ if } \omega \in h^{-1}(m) && h_n'(\omega) = h'(\omega) - \varepsilon_n \text{ if } \omega \in h'^{-1}(m) \\ &h_n(\omega) = h(\omega) + \varepsilon_n \text{ if } \omega \in h^{-1}(l) && h_n'(\omega) = h'(\omega) + \varepsilon_n \text{ if } \omega \in h'^{-1}(l) \\ &h_n(\omega) = h(\omega) \text{ otherwise} && h_n'(\omega) = h'(\omega) \text{ otherwise} \end{aligned}$$

and so $h_n \sim h_n'$ for all n. Since h_n, h_n' converges uniformly to h, h' respectively with $|h_n(\Omega)| = |h(\Omega)|$ and $|h_n'(\Omega)| = |h'(\Omega)|$ for all n. Axiom 5.1 implies $h \sim h'$. Therefore, for all $h, h' \in H$, $h \sim h'$ if $\max h(\Omega) = \max h(\Omega)$ and $\min h(\Omega) = \min h(\Omega)$. And so for all $Y \in \mathcal{X}$ and all $c \in X^0$, $U(Y, c) = U(\{\min_{y \in Y}, \max_{y \in Y}\}, c)$. ■

Proof of Proposition 3.

By (2) of definition 10, $\mu(R \cap R_\sigma^Q) \neq \mu(R' \cap R_\sigma^Q)$ implies $\mu_\sigma(R \cap R_\sigma^Q | s) \neq \mu_\sigma(R' \cap R_\sigma^Q | s)$. Together with (1), $\mu(R \cap R_\sigma^Q) \neq \mu(R' \cap R_\sigma^Q)$ implies $\mu_\sigma(R \cap R_\sigma^Q, s) \neq \mu_\sigma(R' \cap R_\sigma^Q, s)$. The contrapositive is

$$\mu_\sigma(R \cap R_\sigma^Q, s) = \mu_\sigma(R' \cap R_\sigma^Q, s) \Rightarrow \mu(R \cap R_\sigma^Q) = \mu(R' \cap R_\sigma^Q) \qquad (*)$$

Let $(\mu_\sigma(\cdot, s), \mu_\sigma(\cdot \mid s))_{s \in S}$ be a conditional pseudo-belief system. Note that $\mu_\sigma(R, s) = 0$ whenever, $R \cap R_\sigma^s = \emptyset$. Thus condition 1 implies for all $s \in S$ and $R \subset R_\sigma^{\{s\}}$,

$$\mu_\sigma(R \cap R_\sigma^s, s) = \mu(R \cap R_\sigma^{\{s\}})$$

Given $s_1, s_2 \in S$, let $Q = \{s_1, s_2\}$ and (1) implies

$$\mu_\sigma(R_\sigma^Q, s_1) + \mu_\sigma(R_\sigma^Q, s_2) = \mu(R_\sigma^Q)$$

Since both $\mu_\sigma(R_\sigma^Q, s_1)$ and $\mu_\sigma(R_\sigma^Q, s_2)$ are non-negative, there is $\lambda^Q \in \Delta(Q)$, such that $\mu_\sigma(R_\sigma^Q, s_1) = \lambda_{s_1}^Q \mu(R_\sigma^Q)$ and $\mu_\sigma(R_\sigma^Q, s_2) = \lambda_{s_2}^Q \mu(R_\sigma^Q)$. Let $R \subset R_\sigma^Q$ with $\mu_\sigma(R \cap R_\sigma^Q, s_1) = \frac{1}{2}\mu_\sigma(R_\sigma^Q, s_1)$, (*) implies $\mu(R \cap R_\sigma^Q) = \frac{1}{2}\mu(R_\sigma^Q)$, and so $\mu_\sigma(R \cap R_\sigma^Q, s_1) = \lambda_{s_1}^Q \mu(R \cap R^Q)$. Then we have $\mu_\sigma(R \cap R_\sigma^Q, s_2) = (1 - \lambda_{s_1}^Q)\mu(R \cap R^Q) = \lambda_{s_2}^Q \mu(R \cap R^Q)$. Similarly, we have for all $R \subset R_\sigma^Q$ and all rationals r with $\mu_\sigma(R \cap R_\sigma^Q, s_1) = r\mu_\sigma(R_\sigma^Q, s_1)$, $\mu_\sigma(R \cap R_\sigma^Q, s_1) = \lambda_{s_1}^Q \mu(R \cap R^Q)$ and $\mu_\sigma(R \cap R_\sigma^Q, s_2) = \lambda_{s_2}^Q \mu(R \cap R^Q)$. Convexity of μ implies that for all $R \subset R_\sigma^Q$, $\mu_\sigma(R \cap R_\sigma^Q, s_1) = \lambda_{s_1}^Q \mu(R \cap R^Q)$ and $\mu_\sigma(R \cap R_\sigma^Q, s_2) = \lambda_{s_2}^Q \mu(R \cap R^Q)$.

For $|Q| > 2$, we construct in the same way as for $|Q| = 2$ and so for all $R \subset R^* \in \{R_\sigma^Q : Q \in \mathcal{S}\}$, there is an $\lambda^Q \in \Delta(Q)$, such that for all $s \in S$,

$$\mu_\sigma(R, s) = \lambda_s^Q \mu(R \cap R_\sigma^Q).$$

Given $R \in \mathbf{R}$, $R = \bigcup_{Q \in \mathcal{S}:\ s \in Q} R \cap R_\sigma^Q$ and so

$$\mu_\sigma(R, s) = \sum_{Q \in \mathcal{S}:\ s \in Q} \lambda_s^Q \mu(R \cap R_\sigma^Q).$$

Proof of Theorem 4.

1. implies 2. We shall the show the contrapositive holds, namely, if there exists a measurable event in $\widehat{R}$ in $\mathbf{R}$ that is not decomposable then we can find a decision problem d with conditional preference maximizing behavior $b^* = (f_1, \cdots, f_n) \in A^n$ for which

$$W\left(\sum_{s \in S} \mu_\sigma(\Omega, s)\, \mu_\sigma(\mathbf{f}_s^{-1}(\cdot)|s)\right) < W\left(\sum_{s \in S} \mu_\sigma(\Omega, s)\, \mu_\sigma(\widehat{\mathbf{f}}_s^{-1}(\cdot)|s)\right)$$

for some possible signal-contingent action choices $\widehat{b} = (\widehat{f}_1, \cdots, \widehat{f}_n) \in A^n$.

So suppose there exists an event $\widehat{R}$ in $\mathbf{R}$, that is not decomposable. From statement 1 of Lemma 6, it follows there exists a pair of acts f and $\widehat{f}$ for which $W(\mu(\widehat{R})L_{\widehat{R}}^f + (1 - \mu(\widehat{R}))L_{\Omega\setminus\widehat{R}}^{\widehat{f}}) > W(L^{\widehat{f}})$ and $W(\mu(\widehat{R})L_{\widehat{R}}^{\widehat{f}} + (1 - \mu(\widehat{R}))L_{\Omega\setminus\widehat{R}}^f) > W(L^{\widehat{f}})$ but $W(L^{\widehat{f}}) \geqslant W(L^f)$.

Since W exhibits degenerate outcome-lottery independence and mixture continuity, and μ is convex-valued, there exists a sufficiently small decomposable event $R' \in \mathbf{R}$ that is a subset of R' such that for the act $f' := (m)_{R'}\widehat{f}$ we have $f' \succ \widehat{f}$, $f_{\widehat{R}}f' \succ f$ and $f'_{\widehat{R}}f \succ f'$.

So consider the decision problem $d = \langle(\{1, 2\}, \sigma), \{f, f'\}\rangle$, where $\sigma(\omega) = 1$ if $\omega \in \widehat{R}$ and

$\sigma(\omega) = 2$ if $\omega \notin \widehat{R}$. As

$$W(\mu(\widehat{R})\mu_\sigma(\mathbf{f}^{-1}(\cdot)|1) + (1-\mu(\widehat{R}))\mu_\sigma(\mathbf{f}'^{-1}(\cdot)|2)) > W(\mu(\mathbf{f}'^{-1}(\cdot)),$$

$$W(\mu(\widehat{R})\mu_\sigma(\mathbf{f}'^{-1}(\cdot)|1) + (1-\mu(\widehat{R}))\mu_\sigma(\mathbf{f}^{-1}(\cdot)|2)) > W(\mu(\mathbf{f}'^{-1}(\cdot)),$$

and $b^*(1) = b^*(2) = f$. However, since $W(\mu(\mathbf{f}'^{-1}(\cdot)) > W(\mu_\sigma(\mathbf{f}^{-1}(\cdot)|1))$, we have the behavior $\widehat{b}(1) = \widehat{b}(2) = f'$ leads to an ex ante outcome-set lottery that dominates the one generated by the conditional-preference maximizing behavior b^*. Thus we have established b^* is not dynamically consistent. □

2. implies 1. We first consider the class of unambiguous signal problems, namely ones in which $\sigma^{-1}(s) \in \mathbf{R}$ for all $s \in S$. In this case, each feasible behavior b corresponds to an act in F. With slight abuse of our notation, let b also denote this act. That is, $b(\omega) = f(\omega)$ whenever $b(s) = f$. Let $F(d)$ denote the set of acts corresponding to some feasible behavior in b in the unambiguous signal problem d. Since for each $f \in F(d)$ and each (decomposable) event $\sigma^{-1}(s)$ we have $b \succsim f_{\sigma^{-1}(s)}b$ the result follows as a corollary to the following lemma.

Lemma 14 *Fix a finite partition of decomposable events of the state space* $\{R_1, \ldots, R_n\}$*. If* $f \succsim g_{R_i}f$ *for all* $i = 1, \ldots, n$ *then* $f \succsim g$.

Proof. We proceed by induction. Set $h^k := g_{R_1\cup\ldots\cup R_k}f$, so that $h^1 = g_{R_1}f$ and $h^n = g$. As an induction hypothesis, suppose that $f \succsim h^k$. By assumption this hypothesis holds for $k = 1$. For any $k \in \{1, \ldots, n-1\}$ we have $f \succsim h^{k+1}_{R_{k+1}}f = g_{R_{k+1}}$ (by hypothesis) and $f \succsim f_{R_{k+1}}h^{k+1} = h^k$ (by the induction hypothesis). Since W exhibits degenerate singleton outcome-lottery independence and mixture continuity, statement 2 of lemma 6 also holds here, that is, for any decomposable event R and any pair of acts $\widehat{f}$ and $\widehat{g}$, $\widehat{f} \succsim \widehat{f}_R\widehat{g}$ and $\widehat{f} \succsim \widehat{g}_R\widehat{f}$ implies $\widehat{f} \succsim \widehat{g}$. Hence we have established $f \succsim h^{k+1}$, as required. ■

To complete the proof, we show that for each decision problem $d = \langle (S, \sigma), A\rangle$ there exists an *unambiguous equivalent* problem $\widehat{d} = \langle (S, \widehat{\sigma}), A\rangle$ that satisfies for each $s \in S$, every $Q \in \mathcal{S}$ with $s \in Q$ and every $\lambda^Q \in \Delta(Q)$:

1. $\mu\left(\widehat{\sigma}^{-1}(s) \cap R^Q_\sigma\right) = \lambda^Q_s \mu(R^Q_\sigma)$
2. $\mu_{\widehat{\sigma}}(\mathbf{f}^{-1}(\cdot)|s) = \mu_\sigma(\mathbf{f}^{-1}(\cdot)|s)$, for all $f \in A$.

Notice that this means that for any behavior b in the decision problem d the corresponding behavior $\widehat{b}$ in which $\widehat{b}(s) \equiv b(s)$ induces the same ex ante and conditional outcome-set lotteries. Hence dynamic consistency of behavior in d will follow from dynamic consistency of behavior in the unambiguous equivalent problem $\widehat{d}$.

To construct the unambiguous equivalent problem $\widehat{d}$, we first observe that since μ is convex-ranged, it follows that for each $Q \in \mathcal{S}$ we can divide the measurable event R_σ^Q into a $|Q|$-element measurable partition $\{R_\sigma^{Q,s}\}_{s \in Q}$. That is, each $R_\sigma^{Q,s}$ is in $\mathbf{R}$ and has measure $\lambda_s^Q \mu\left(R_\sigma^Q\right)$. We now demonstrate that this can be done in such a way that ensures $\mu_{\widehat{\sigma}}(\mathbf{f}^{-1}(\cdot)|s) = \mu_\sigma(\mathbf{f}^{-1}(\cdot)|s)$ for all $f \in A$ and all $s \in Q$. Consider the (finite) measurable partition Π^Q of R_σ^Q where each element of this partition can be expressed as

$$R_\sigma^Q \cap \bigcap_{f \in A} R_f^{Y_f},$$

for some $Y_f \in \mathcal{X}^{f(R_\sigma^Q)^+}$, for each $f \in A$. Again since μ is convex-ranged, we can divide each generic element $\widehat{R}^Q \in \Pi^Q$ into $|Q|$ events and each event has the corresponding measure $\lambda_s^Q \mu\left(R_\sigma^Q\right)$. Denote each such event by $\widehat{R}^{Q,s}$. Then for each $s \in S$ we set $R_\sigma^{Q,s} := \bigcup_{\widehat{R}^Q \in \Pi^Q} \widehat{R}^{Q,s}$.

Finally we set $\widehat{\sigma}(\omega) := s$ whenever

$$\omega \in \bigcup_{Q \in \mathcal{S}:\ s \in Q} R_\sigma^{Q,s}.$$

Notice by construction, $\{\widehat{\sigma}^{-1}(s)\}_{s \in S}$ is a measurable partition of Ω with

$$\mu(\widehat{\sigma}^{-1}(s)) = \sum_{Q \in \mathcal{S}:\ s \in Q} \mu\left(R_\sigma^{Q,s}\right) = \sum_{Q \in \mathcal{S}:\ s \in Q} \lambda_s^Q \mu\left(R_\sigma^Q\right) = \mu_\sigma\left(R_\sigma^s, s\right).$$

Furthermore, for each $f \in A$ and each $Q \in \mathcal{S}$, we have $\mu_{\widehat{\sigma}}(\mathbf{f}^{-1}(\cdot)|s) = \mu_\sigma(\mathbf{f}^{-1}(\cdot)|s)$ as desired.

■

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