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Bayes and Hurwicz without Bernoulli ☆

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Abstract

We provide a theory of decision under ambiguity that does not require expected utility maximization under risk. Instead, we require only that a decision maker be probabilistically sophisticated in evaluating a subcollection of acts. Three components determine the decision maker's ranking of acts: a prior, a map from ambiguous acts to equivalent risky lotteries, and a generalized notion of certainty equivalent. The prior is Bayesian, defined over the inverse image of acts for which the decision maker is probabilistically sophisticated. Ambiguity preferences are similar to Hurwicz, depending on an act's best- and worst-case interpretations. The generalized certainty equivalent may, but need not, come from a Bernoulli utility. The ability to combine appealing theories of risk and ambiguity at will has been sought after but missing from the literature, and our decomposition provides a promising way forward.

JEL classification: D80; D81

Keywords: Uncertainty; Risk; Ambiguity; Certainty equivalent

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1. Introduction

Decision makers are said to face ambiguity when they do not have a uniquely defined prior over some decision relevant events. For example, climate change projections from different models provide credible but divergent estimates of the probability distribution of climate sensitivity (see the discussion in Heal and Millner, 2014). How a decision maker (DM) responds to ambiguity (as when evaluating climate policies) is a distinct question from how the DM makes calculated risks with known probabilities. For instance, it may be that a DM is calculating and level-headed when assessing climate change but a maniacal fiend when she enters a casino.

Our goal is to isolate the role of ambiguity, enabling researchers to study it without committing to a particular theory of decision under risk. Attitudes toward known risks can be studied in some collection of problems on which the DM is probabilistically sophisticated (Cohen et al., 1987; Machina and Schmeidler, 1992)—that is, on which known probabilities guide her actions, regardless of whether she follows expected utility or some other theory of decision under risk. Whether the DM is Bayesian in forming beliefs is a different question from whether she has a Bernoulli utility function (Machina and Schmeidler, 1995) (or, employing the terminology of Nehring (2007), the DM may be probabilistically-sophisticated without necessarily being utility-sophisticated). Requiring no more than probabilistic sophistication can reassure a researcher that findings about behavior under ambiguity are not artifacts of an underlying assumption about attitudes toward risks.

This is easier said than done: there have been a slew of negative results showing that, under natural assumptions, existing models of decision under ambiguity force the researcher to make a choice. Adopt expected utility theory as the decision model for risk, or commit to a world in which all non-degenerate events are ambiguous. Beginning with Marinacci's (2002) results on the multiple priors model, this finding was shown by Strzalecki (2011) to apply to the variational model (Maccheroni et al., 2006), and generalized by Cerreia-Vioglio et al. (2012) to a wider collection of theories of decision under ambiguity.

We do not wish to interpret these formal results as implying that we cannot model all of the agent types that we would like. It remains desirable to jointly model non-EU risk preferences and common ambiguity preferences, as various attempts show. For example, Nakamura (1995) provides a joint axiomatization of rank dependent utility (regarding risk) and Choquet expected utility (regarding ambiguity). Dean and Ortoleva (2017) propose a possible hedging axiom that could drive non-EU behavior under both risk and ambiguity. Similarly, Lleras et al. (2019) characterize preferences which can equally be interpreted as loss averse (non-EU regarding risk) or as MEU (ambiguity averse).

Success in this direction, however, remains tied to specific theories of decision under both risk and ambiguity. In principle, however, these preference types are independent, and a DM might display any combination of risk and ambiguity attitudes. It is therefore desirable to build a theory which is agnostic regarding both types of attitude, and in which their effects are isolated. This paper aims to provide such a theory: the roles of the DM's prior probability, attitude toward ambiguity, and attitude toward known risk are disentangled. In the spirit of Machina and Schmeidler (1995), we aim to separate Hurwicz (1951) from Bernoulli, while keeping Bayes.

We address each of these pieces in turn. Our first step is to consider a collection of acts thought of as risky. Since the DM is probabilistically sophisticated, she has a prior probability over the different consequences of a risky act. We can therefore work with the inverse image along the prior of each prize, and restrict attention to lotteries with objective probabilities.

	Prob 1/2		Prob 1/2	
	A	B	C	D
f	4	8	4	0
g	4	4	8	0
$\hat{g}$	0	8	4	4
$\hat{f}$	0	4	8	4

Fig. 1. Example based on Machina (2009) including both risk and ambiguity.

We use risky acts as our building blocks for studying ambiguous acts. Following the approach in the expected uncertain utility (EUU) model of Gul and Pesendorfer (2014) (foreshadowed partly in Jaffray, 1989), we replace each ambiguous act with a closed interval of risky acts. The lower and upper bounds on these intervals correspond to the most extreme cases of how the ambiguity might be resolved. The DM's ambiguity preferences are captured by her *unambiguous equivalent* (monetary prize) for each corresponding interval of prizes.[1]

The last piece is the DM's *sure equivalent* function, which reflects her risk preferences. The sure equivalent takes as its input simple lotteries over monetary prizes, built from the prior and the unambiguous equivalent. It maps each such lottery to the prize which the DM views as equivalent.

These three pieces give our main result. Given probabilistic sophistication, we show that if a DM satisfies axioms similar to those of Gul and Pesendorfer (2014), along with a strengthening of Savage's (1954)'s postulate P3 that embodies a notion of (stochastic) independence between risk and ambiguity, then the DM's preferences can be represented with the composition of our three components. This provides a way to generalize Gul and Pesendorfer's (2014) Expected Uncertain Utility (EUU), which we call a Generalized Uncertain Utility (GUU) representation.

A simple example illustrates how these three pieces combine, applying them step-by-step to determine the DM's certainty equivalent. This lets us rank arbitrary acts. The example is related to Machina's (2009) Reflection Effect, which causes problems of one kind or other for most major theories of decision under ambiguity (Baillon et al., 2011).[2] Both risk and ambiguity are present, and by addressing the two separately, we are able to account for common behavior (see L'Haridon and Placido, 2010, for experimental evidence).

Let there be four states of the world, referred to as A, B, C, and D. The events $\{A, B\}$ and $\{C, D\}$ are risky, each occurring with probability 0.5. Beyond these, the DM has no further information about any nontrivial events. In keeping with our motivation, we can imagine that the setting is one of choice given anticipated climate change. The ambiguous event could be whether a particular important species will still be around in 10 years. The risky event could be whether an emerging biotechnology will be viable at the end of the same 10-year period (taking the liberty of assuming that we can measure this).

For concreteness, consider the acts in Fig. 1. The first observation regarding this choice problem is that some popular models of decision under ambiguity require indifference between f and g, and between $\hat{f}$ and $\hat{g}$, while strict preferences are intuitive and common (Machina, 2009; L'Haridon and Placido, 2010). The second important observation is that the choice between f and g is equivalent to the choice between $\hat{f}$ and $\hat{g}$; $\hat{f}$ is a mirror image of f, and similarly $\hat{g}$ is a mirror image of g. However, Machina (2009) notes that shifting 0 to 4 in event D, then shifting 4 down to 0 in event A allows us to obtain act $\hat{g}$ from f and $\hat{f}$ from g. This sequence of common outcome tail shifts suggests that a DM who prefers f to g should prefer $\hat{g}$ to $\hat{f}$. Experimentally,

1 A related approach, viewing ambiguous lotteries as intervals of objective lotteries, is in Hill (2019).

2 An exception is the vector expected utility theory of Siniscalchi (2009).

however, most subjects' choices instead reflect the informational symmetry; here, the preference between f and g would match the preference between $\hat{f}$ and $\hat{g}$ (L'Haridon and Placido, 2010).

Our approach both preserves the indifferences arising from the informational symmetry and allows for strict preferences between f and g (as well as their counterparts). In order to show how our theory works, we show how an agent's underlying ambiguity preferences would yield an act ranking. Act f gives an uncertain prize of $\{4, 8\}$ with probability $1/2$ and an uncertain prize of $\{0, 4\}$ with probability $1/2$. Let $u(x, y)$ be the prize in the set of outcomes X that the DM views as equivalent to the ambiguous prize ranging from x to y (where $x \leqslant y$). For example, suppose that the DM views an ambiguous outcome between x and y as equally desirable as receiving $\sqrt{(x+1)(y+1)} - 1$, similar to the Cobb-Douglas ambiguity preferences Binmore (2009) proposes (see the discussion in Grant et al., 2019). Then we can replace f with a lottery paying $u(4, 8) = \sqrt{45} - 1$ with probability $1/2$ and $u(0, 4) = \sqrt{5} - 1$ with probability $1/2$. Similarly, we can replace g with a lottery paying $u(4, 4) = 4$ with probability $1/2$ and $u(0, 8) = 2$ with probability $1/2$. Now let $s(\cdot)$ be the sure equivalent. We rank the acts f and g by comparing

$$
\begin{aligned}
c(f) \ \left(= c(\hat{f})\right) &= s(0.5, u(4,8); 0.5, u(0,4)) \\
&= s(0.5, \sqrt{45} - 1; 0.5, \sqrt{5} - 1) \approx s(0.5, 5.7; 0.5, 1.2) \\
\text{versus } c(g) \ \left(= c(\hat{g})\right) &= s(0.5, u(4,4); 0.5, u(0,8)) \\
&= s(0.5, 4; 0.5, 2)
\end{aligned}
$$

where $c(h)$ is the certainty equivalent for a given act h.

Clearly there is nothing that requires the certainty equivalents of f and g to be the same in this case. If the DM is close to a risk-neutral expected utility maximizer, she would prefer f to g. But she might prefer g to f either because she is risk averse and an expected utility maximizer, or for many reasons that would be consistent with non-EU decisions under risk: g has a higher security level (Gilboa, 1988; Jaffray, 1988; Cohen, 1992), it may appeal to disappointment averse decision makers (see Gul, 1991), and so on.

A final feature of our approach is worth noting here. A concern about models involving both ambiguity and known risks is that it may be possible for a DM to hedge ambiguity (see Raiffa, 1961). One way to address this is to relax the requirement that risky events form a σ-algebra, instead imposing only that they form a λ-system (Epstein and Zhang, 2001). This idea is taken up in the context of probabilistic sophistication by Chew and Sagi (2008), and is the approach we follow here.

A formal definition of a λ-system appears in Section 2, but at this stage it is enough to note that it is similar to a σ-algebra but with the important difference that it is required to be closed only under complements and countable disjoint unions; it need not be closed under intersections. For instance, we can imagine a DM who is concerned about climate change and faces decisions related to the weather in Canberra. The following example shows in this context why we should not expect the set of risky events to be closed under intersections.

Example. A DM is uncertain about both the average maximum temperature and the monthly rainfall in Canberra in June next year. She knows that Canberra's historic mean average maximum temperature for June is 7°C and its historic mean rainfall for June is $19mm$. She also knows that average monthly maximum temperature is distributed symmetrically around its historic mean while rainfall is not. However, she cannot recall whether the distribution of rainfall is positively or negatively skewed. Let $\overline{T} \geqslant 7$ (respectively, $RF \geqslant 19$) denote the event that Canberra's average maximum temperature (respectively, rainfall) in June is greater than or equal to its historic mean, and let $\overline{T} < 7$ (respectively, $RF < 19$) denote the complementary event. Letting p denote the unknown probability that the event $RF \geqslant 19$ will obtain, the joint distribution over these four events is

	$RF \geqslant 19$	$RF < 19$
$\overline{T} \geqslant 7$	$p/2$	$(1-p)/2$
$\overline{T} < 7$	$p/2$	$(1-p)/2$

Notice that the event "June's average maximum temperature is above its historic mean" and the event "either both June's average maximum temperature and its rainfall are above their respective historic means or both are below"

$$= \left((RF \geqslant 19) \cap \left(\overline{T} \geqslant 7\right)\right) \cup \left((RF < 19) \cap \left(\overline{T} < 7\right)\right)$$

are both risky (with probability $1/2$). However, their intersection, the event "both June's average maximum temperature and its rainfall are above their respective historic means"

$$= (RF \geqslant 19) \cap \left(\overline{T} \geqslant 7\right)$$

has unknown probability and hence is not risky. That is, the set of risky events is not closed under intersection although casual inspection reveals that the above collection of risky events is closed under complements and disjoint unions. A λ-system is thus the more appropriate structure to represent the collection of risky events.

The rest of the paper is organized as follows. We begin with the basic set-up in Section 2. We then present in Section 3 the formal definition of the family of Generalized Uncertain Utility maximizers which is followed in Section 4 with our set of axioms and representation theorem. Section 5 provides one way to endogenize the collection of risky events that were taken as given in our axiomatization. We achieve this by employing Epstein and Zhang's (2001) behavioral definition for classifying *unambiguous* events. In Section 6 we present a finite state version of the model. We conclude in Section 7 with two examples that demonstrate the two broad ways in which the family of GUU maximizers extends Gul and Pesendorfer's (2014) family of EUU maximizers. Unless stated otherwise, proofs appear in the appendix.

2. Preliminaries

We employ Gul and Pesendorfer's (2014) version of a Savage (1954) setting of purely subjective uncertainty. The set of final prizes $X = [\ell, m]$, $\ell < m$ is a non-degenerate interval of the real line. Let Ω be the state space. For any pair of events $A, B \subseteq \Omega$, $B \backslash A$ denotes the set of elements that are in B but not in A. Hence $\Omega \backslash A$ denotes the complement of A. The objects of choice, denoted by $\mathcal{F}$, are elements of the set of simple acts, that is, mappings $f : \Omega \to X$ with finite range. The decision maker (hereafter, DM) is characterized by her preferences over acts, a binary relation $\succsim$ on $\mathcal{F}$.

We identify any $x \in X$ with the (constant) act f in which $f(\omega) = x$ for all ω. For any pair of acts f and g in $\mathcal{F}$ and any event $A \subset \Omega$, we write $f_A g$ for the act that agrees with f on A and with g on $\Omega \backslash A$.

In addition to the primitives Ω, X, and $\succsim$, we fix a collection $\mathcal{R}$ of subsets of Ω to denote the set of *risky* events. These are the events to which the DM can assign a unique prior. A subset of the risky events are the null events to which the DM can assign zero weight. Formally, an event N is *null* if $f \sim g_N f$ for all $f, g \in \mathcal{F}$. We denote by $\mathcal{N}$ the set of null events.

In view of the arguments presented in Epstein and Zhang (2001) and illustrated by the example from Section 1 above, it is natural that the class of risky events for a DM need only comprise a λ-system (also known as a Dynkin system) of subsets of the state space rather than a full-blown σ-algebra. That is, assume $\mathcal{R}$ satisfies the following three conditions:

$R.1$ $\Omega \in \mathcal{R}$;
$R.2$ $R \in \mathcal{R}$ implies $\Omega \backslash R \in \mathcal{R}$ (that is, $\mathcal{R}$ is closed under complements); and
$R.3$ $R_n \in \mathcal{R}$, $n = 1, 2, \ldots$ and $R_n \cap R_m = \varnothing$ for all $n \neq m$, implies $\bigcup_{n=1}^{\infty} R_n \in \mathcal{R}$ (that is, $\mathcal{R}$ is closed under countable disjoint unions).

In the sequel, it will sometimes be convenient to invoke two additional conditions that a λ-system satisfies, as the following lemma shows (for a proof see Epstein and Zhang, 2001, p. 270).

Lemma 1 *(Epstein and Zhang's (2001) Lemma 2.1). If the collection of risky events $\mathcal{R}$ satisfies conditions $R.1 - R.3$ then it also satisfies:*

$R.4$ *$R, R' \in \mathcal{R}$ and $R \subseteq R' \implies R' \backslash R \in \mathcal{R}$ (that is, $\mathcal{R}$ is closed under relative complements); and*
$R.5$ *$R_n \in \mathcal{R}$ and $R_n \subseteq R_{n+1}$, $n = 1, 2, \ldots, \implies \bigcup_{n=1}^{\infty} R_n \in \mathcal{R}$ (that is, $\mathcal{R}$ is closed under taking the union of an increasing, in terms of set inclusion, sequence of sets).*

We call an act g *risky* if it is measurable with respect to $\mathcal{R}$, that is, $g^{-1}(x) \in \mathcal{R}$ for all $x \in X$. Let $\mathcal{G} \subset \mathcal{F}$ denote the set of risky acts.

Following Gul and Pesendorfer's (2014) approach, we suppose the DM uses elements of non-null risky events (that is, $R \in \mathcal{R} \setminus \mathcal{N}$) to quantify the uncertainty of any event. In particular, if two events contain exactly the same elements of $\mathcal{R} \setminus \mathcal{N}$ and the same is true of their respective complements then the DM should be indifferent between betting on either of these events. In light of this, a maximally ambiguous event is one for which it and its complement contain no element of $\mathcal{R} \setminus \mathcal{N}$. Adopting the terminology of Gul and Pesendorfer (2014), we will refer to such a maximally ambiguous event as *diffuse*. More formally, an event $D \subset \Omega$ is *diffuse* if, for every non-null risky event $R \in \mathcal{R} \setminus \mathcal{N}$, we have both $D \cap R \neq \varnothing$ (that is, D intersects R) and $(\Omega \setminus D) \cap R \neq \varnothing$ (the complement of D intersects R). We denote by $\mathcal{D}$ the set of diffuse events.

We say an act h is *diffuse* if its inverse image generates a diffuse partition of Ω, that is, $h^{-1}(x) \in \mathcal{D}$ for all $x \in X$ such that $h^{-1}(x) \neq \varnothing$. Let $\mathcal{H} \subset \mathcal{F}$ denote the set of diffuse acts.

3. Generalized uncertain utility maximization

We begin our description of a Generalized Uncertain Utility (GUU) maximizer with the definition of a prior, a probability measure that quantifies precisely the uncertainty associated with risky events.

Definition 1 *(Prior).* A *prior* π is a countably-additive and convex-ranged probability measure defined on the set of risky events $\mathcal{R}$.

Countable additivity is defined in the usual way. Convex-ranged means that for any $A \in \mathcal{R}$ and any $r \in (0, 1)$ there exists a risky event $B \in \mathcal{R}$ such that $B \subset A$ for which $\pi(B) = r\pi(A)$.[3] If we associate a prior π with a DM's preference relation $\succsim$, then we may view the restriction of $\succsim$ to the set of risky acts $\mathcal{G}$ as reflecting the DM's risk preferences. We use the prior π to map each risky act in $\mathcal{G}$ to a lottery, as follows: let $\Delta_0(X)$ denote the set of simple lotteries defined on X. That is, each $L \in \Delta_0(X)$ is a finite ranged function $L : X \to [0, 1]$ satisfying $\sum_{x \in X} L(x) = 1$. For each $f \in \mathcal{G}$, let $\pi \circ f^{-1}$ denote the lottery $L \in \Delta_0(X)$ in which $L(x) = \pi \circ f^{-1}(x)$. In order for the restriction of $\succsim$ to $\mathcal{G}$ to fully characterize the DM's risk preferences, the range of the mapping $f \mapsto \pi \circ f^{-1}(x)$ should be $\Delta_0(X)$.

Lemma 2 establishes that this is indeed the case. The proof uses the convex-ranged property of the prior.

Lemma 2. *Fix a prior π. Then, for each lottery $L \in \Delta_0(X)$ there exists an act $f \in \mathcal{G}$ such that $\pi \circ f^{-1} = L$.*

A natural way to represent a DM's risk preferences over $\Delta_0(X)$ is by associating each lottery with its sure equivalent, that is, with the (for sure) prize which the DM would view as equally desirable as the lottery. Since we wish to allow for risk preferences that can accommodate Allais-type paradoxes, we follow Epstein and Zhang (2001) and take no stand on what properties sure equivalents must have apart from requiring them to exhibit a standard form of (stochastic) monotonicity and (mixture) continuity.

For each $x \in X$, let $[x]$ denote the degenerate lottery that yields x with probability 1.[4] The sure equivalent of a lottery is formally expressed as follows.

Definition 2 *(Sure Equivalent).* A *sure equivalent* is a monotonic and mixture continuous function $s\colon \Delta_0(X) \to X$ satisfying $s([x]) = x$ for all x in X. *Monotonicity* requires respecting the (partial) ordering of first-order stochastic dominance. That is, $s(L) \geqslant s(L')$ whenever

$$\sum_{y \leqslant x} L(y) \leqslant \sum_{y \leqslant x} L'(y) \text{ for all } x \in X\,,$$

with $s(L) > s(L')$ if the inequality above holds strictly for some $x \in X$. *Mixture continuity* requires that for any three lotteries $L, L', L'' \in \Delta(X)$ the sets

$$\left\{\gamma \in [0,1] : s\left(\gamma L + (1 - \gamma L')\right) \geqslant s(L'')\right\} \text{ and}$$
$$\left\{\gamma \in [0,1] : s\left(\gamma L + (1 - \gamma L')\right) \leqslant s(L'')\right\}$$

are closed.

Although we expressly wish to allow for risk preferences that do not conform to expected utility theory, if the DM were an expected utility maximizer (over risky acts), then that would imply the existence of an increasing Bernoulli utility index $V : X \to \mathbb{R}$ such that

$$s(L) = V^{-1}\left(\sum_{x \in X} L(x)V(x)\right).$$

[3] Notice that this requires the cardinality of the state space Ω is at least that of the continuum and that the set of null events $\mathcal{N}$ is complete.

[4] That is, $[x](x) = 1$ and, for every $y \neq x$, $[x](y) = 0$.

To accommodate Allais-type behavior, Machina and Schmeidler (1992) introduce the concept of a probabilistically sophisticated DM whose preferences over risky acts in the current setting can be characterized by a prior and a sure equivalent:

Definition 3 *(Probabilistic Sophistication).* The preference relation $\succsim$ is *probabilistically sophisticated* with respect to the set of risky events $\mathcal{R}$ if there exists a prior π and a sure equivalent s such that for every pair of acts f and g in $\mathcal{G}$,

$$f \succsim g \iff s\left(\pi \circ f^{-1}\right) \geqslant s\left(\pi \circ g^{-1}\right).$$

We initially take probabilistic sophistication as our starting point, but we provide in Section 5 a purely subjective (that is, preference-based) derivation of the set of risky events based on Epstein and Zhang's (2001) axiomatization.[5]

In order to evaluate an act that is not risky, we assume (analogous to Gul and Pesendorfer, 2014) that the DM constructs a greatest lower-bound risky act and a least upper-bound risky act that together form what we refer to as a *risky envelope* for that act. Moreover, we require that every element of the coarsest common refinement of the two partitions induced by the two bounds is in $\mathcal{R}$. We define these bounds as follows.

Definition 4 *(Risky Envelope).* Fix an act $f \in \mathcal{F}$. Say that $\left[\underline{f}, \bar{f}\right]$ with $\underline{f}, \bar{f} \in \mathcal{R}$ constitutes a *risky envelope* of f if $\underline{f} \leqslant f \leqslant \bar{f}$ and, for any risky act $g \in \mathcal{G}$:

(i) $f \geqslant g \implies \left\{\omega \in \Omega\colon \underline{f}(\omega) < g(\omega)\right\}$ is null;
(ii) $f \leqslant g \implies \left\{\omega \in \Omega\colon \bar{f}(\omega) > g(\omega)\right\}$ is null;
(iii) $\underline{f}^{-1}(x) \cap \bar{f}^{-1}(y) \in \mathcal{R}$, for all x, y in X.

Below, we refer to $\underline{f}$ (respectively, $\bar{f}$) as a *risky lower-sleeve* (respectively, *risky upper-sleeve*) of f. As the following lemma establishes, the concept of a *risky envelope* is well-defined for a probabilistically sophisticated preference relation.

Lemma 3. *Fix a probabilistically sophisticated preference relation $\succsim$. Then every act $f \in \mathcal{F}$ has a risky envelope.*

In particular, it turns out that for any diffuse act $h \in \mathcal{H}$, its risky-envelope $\left[\underline{h}, \bar{h}\right]$ comprises two constant acts x and y, where x, its risky lower-sleeve, is defined as

$$x = \min\{z \in X: h^{-1}(z) \neq \varnothing\}$$

and y, its risky upper-sleeve, is defined as

$$y = \max\{z \in X: h^{-1}(z) \neq \varnothing\}.$$

[5] Kopylov (2007) formulates a characterization of probabilistic sophistication in an even more general setting, where risky events form a mosaic (see also Abdellaoui and Wakker, 2005). In particular, a mosaic need not be closed under disjoint unions. Moreover, Kopylov's representation result shows that the agreeing probability need only be finitely additive and in general need not be convex ranged. This is too general a setting for us, as our Lemma 2 would not hold, preventing us from recovering the decision-maker's full risk preferences.

That is, its risky envelope is an interval of prizes $[x, y]$.

Mirroring the treatment of sure equivalents above, a natural way to represent a DM's preferences over intervals of prizes is by associating each interval with its unambiguous equivalent, that is, with the unambiguous (for sure) prize which the DM would view as equally desirable as the interval. The unambiguous equivalent of an interval of prizes is formally expressed as follows.

Definition 5 *(Unambiguous Equivalent).* An *unambiguous equivalent* is a monotonic function $u\colon \{(x, y) \in X \times X : x \leqslant y\} \longrightarrow X$ satisfying $u(x, x) = x$ for all x in X.

We now have all the ingredients needed to state the formal definition of a Generalized Uncertain Utility (GUU) maximizer.

Definition 6 *(The Class of GUU Maximizers).* A preference relation $\succsim$ is a member of the class of *Generalized Uncertain Utility maximizers* if there exists a triple $\langle \pi, u, s\rangle$, where π is a prior, u is an unambiguous equivalent, and s is a sure equivalent, such that $\succsim$ is represented by the certainty equivalent $c(f) = s\left(\pi \circ u\left(\underline{f}, \bar{f}\right)^{-1}\right)$, where $\pi \circ u\left(\underline{f}, \bar{f}\right)^{-1}$ is the lottery given by

$$\pi \circ u\left(\underline{f}, \bar{f}\right)^{-1}(x) = \pi\left(\left\{\omega \in \Omega : u\left(\underline{f}(\omega), \bar{f}(\omega)\right) = x\right\}\right).$$

4. Characterization

Turning now to the characterization, we begin with four axioms from Gul and Pesendorfer (2014) that we dub *Ordering*, *Monotonicity*, *Continuity* and *Diffuse-Event Exchangeability*, respectively.

Axiom 1 *(Ordering). The binary relation $\succsim$ is complete and transitive.*

Axiom 2 *(Monotonicity). For any pair of acts $f, g \in \mathcal{F}$, if $f > g$, then $f \succ g$.*

Axiom 3 *(Continuity). For any sequence of acts $\{f_n\}$ and any pair of acts g and h, if $g \succsim f_n \succsim h$ for all n and f_n converges uniformly to f, then $g \succsim f \succsim h$.*

Axiom 4 *(Diffuse-Event Exchangeability). For any pair of consequences $x, y \in X$, any non-null risky event $R \in \mathcal{R}\backslash\mathcal{N}$ and any pair of diffuse events D, D' in $\mathcal{D}$,*

$$x_{D\cap R}\, y \sim x_{D'\cap R}\, y\,.$$

Ordering and *Monotonicity* are standard. Gul and Pesendorfer (2014) note that *Continuity* is what would be required to get a continuous Bernoullian utility index when proving Savage's theorem in a setting with real-valued prizes. Here it serves a similar role to ensure that the unambiguous equivalent is continuous.

Diffuse Event Exchangeability requires the DM to be indifferent between betting on (or against) a pair of events $R \cap D$ and $R \cap D'$ whenever R is risky and both D and D' are diffuse. To motivate this axiom, Gul and Pesendorfer point out that this accords with their main hypothesis of a DM using non-null risky events to calibrate the events she views as ambiguous. Notice that neither $R \cap D$ nor $R \cap D'$ contain any non-null risky events while their respective complements contain exactly the same collection of risky events. Diffuse-Event Exchangeability is thus an implication of this hypothesis.

Intuitions may pull in different directions regarding this axiom. In support of Diffuse Event Exchangeability, recall the exchange between Jevons and Keynes regarding how to assign probabilities to events about which we are radically uncertain. Consider the events that "a Platythliptic Coefficient is positive," "a Platythliptic Coefficient is a perfect cube," and "a Platythliptic Coefficient is allogeneous" (Jevons 1877, pg. 212, Keynes 1921, pg. 46). Diffuse events are to be understood as at least as mysterious as these, and the axiom reflects the intuition that the agent will not prefer to bet on one rather than another (the possible intuition that positive numbers are a priori more likely than perfect cubes notwithstanding). Intuitions may pull in the opposite direction if we consider the events "a Platythliptic Coefficient is between 0 and 5" and "a Platythliptic Coefficient is between 0 and 100." Since the first event is a subset of the second, it may seem to have a smaller probability, and it may therefore seem wise to prefer to bet on a risky event conjoined with the second rather than with the first.[6] If so, this example would be a counterexample to Diffuse Event Exchangeability. For present purposes we reject this counterexample, for the following reason: the intuition that the second event is larger than the first is a geometric intuition; we can imagine the second event taking up far more space if we were to draw the possibilities. Our framework is a measure-theoretic one, however, and so probabilities are not evaluated according to a geometric conception of size. Furthermore, it is known that these two ways of evaluating size can come apart in surprising ways (see e.g. Banach and Tarski 1924, Oxtoby 1980, pg. 8). Hence, we argue that the first kind of intuition is the relevant one here. Exploring ambiguity attitude from a geometric perspective might well be interesting, but is beyond the scope of this paper.

The next axiom, a strengthening of Savage's postulate P3, applies to pairs of acts that differ only on a risky event, and where conditional on that risky event obtaining, each act has a (maximally) ambiguous bet based on whether or not a particular diffuse event obtains.[7] It embodies the idea that the preference a DM expresses between such a pair of ambiguous bets is (stochastically) independent of "risk" by requiring the conditional preference to be governed by the DM's *unconditional preference*, irrespective of what might arise should the conditioning risky event not obtain. Recall that by definition, each diffuse event intersects with every non-null risky event. Thus the realization of a risky event might reasonably be viewed by the DM as providing her with no information about bets based on diffuse events.

Axiom 5 *(Conditional Diffuse-Bet Independence). For any four prizes x, x^*, y, y^* in X, any act $f \in \mathcal{F}$, any non-null risky event $R \in \mathcal{R} \setminus \mathcal{N}$, and any pair of diffuse events D, $\widehat{D} \in \mathcal{D}$,*

$$\left(x^*_D x\right)_R f \succsim (y^*_{\widehat{D}} y)_R f \Longleftrightarrow x^*_D x \succsim y^*_{\widehat{D}} y\,.$$

In conjunction with the first four axioms, Axiom 5 provides us with a characterization of the class of Generalized Uncertain Utility maximizers, our main representation result.

Theorem 4. *Suppose the preference relation $\succsim$ is probabilistically sophisticated.*
Then the following are equivalent.

[6] We thank an anonymous referee for pressing us regarding an example of this type.

[7] Savage's P3 corresponds to the case in which $x = x^*$ and $y = y^*$.

1. *The preference relation $\succsim$ satisfies Axioms 1–5.*
2. *The preference relation $\succsim$ admits a GUU representation.*

5. Endogeneously determined risk

In the foregoing, the set $\mathcal{R}$ of risky events is taken as given. This may be unobjectionable in some circumstances—for example in a casino—but in other circumstances agents may differ in which events they see as risky and which as ambiguous. In such cases, it is desirable to acknowledge the subjective aspect of ambiguity, and to infer which events the DM views as risky from their behavior if possible. We do not rule out this approach: in this section, we provide a way to endogenize the set of events that the decision-maker views as unambiguous. We do so by utilizing Epstein and Zhang's (2001) behavioral definition for classifying unambiguous events. We discuss the prospects for endogenously-defined risk at the end of the section.

Definition 7 *(Unambiguous Events).* An event R is *unambiguous* if:

(a) for all subsets B of $\Omega \setminus R$, all outcomes x, y, z, z' in X, and all acts $f, g, h \in \mathcal{F}$, such that $f^{-1}(\{x, y\}) = g^{-1}(\{x, y\}) = \Omega$,

$$f_B z_R h \succsim g_B z_R h \implies f_B z'_R h \succsim g_B z'_R h; \text{ and,}$$

(b) the condition obtained if in (a) the event R is replaced everywhere by $\Omega \setminus R$ is also satisfied.

Otherwise R is *ambiguous*. We denote by $\mathcal{U}$ the set of unambiguous events.

Since the defining invariance condition holds both for $R = \varnothing$ (with $\Omega \setminus R = \Omega$) and for $R = \Omega$ (with $\Omega \setminus R = \varnothing$) we see that both $\varnothing$ and Ω are naturally classified as unambiguous. Similarly, every null event N in $\mathcal{N}$ and its complement $\Omega \setminus N$ are also classified as unambiguous. More generally, as Epstein and Zhang (2001) note, the requirement that the defining invariance condition holds for both an event and its complement builds into the definition the intuitive feature that an event is classified as unambiguous if and only if its complement is.

To provide insight as to why their defining invariance condition is the appropriate one for determining when an event should be deemed unambiguous, Epstein and Zhang (2001) begin by pointing out that the first two acts being compared yield identical outcomes if a state outside of B is realized. So the comparison may be viewed as one between bets conditional on B with stakes x and y and outcomes shown for $\Omega \setminus B$. If $x > y$ then the indicated ranking reveals that the DM views the event $f^{-1}(x) \cap g^{-1}(y) \cap B$ as conditionally more likely than the (disjoint) event $f^{-1}(y) \cap g^{-1}(x) \cap B$.[8] Hence, the key invariance property is that this conditional likelihood ranking be unaffected by changing the outcome from z to z' on the putatively unambiguous R. Epstein and Zhang go on to highlight that the constancy of the acts being compared on the putatively unambiguous R is natural since the definition has been designed to reveal that R in its entirety is unambiguous and need not (and, indeed, should not) imply anything about the ambiguity status of its subsets.

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[8] Notice that the other two (disjoint) subsets of B, $f^{-1}(x) \cap g^{-1}(x) \cap B$ and $f^{-1}(y) \cap g^{-1}(y) \cap B$ yield the same outcome for the two acts being compared.

We shall call an act *unambiguous* if it is measurable with respect to $\mathcal{U}$, that is, $f^{-1}(x) \in \mathcal{U}$ for all $x \in X$. At the risk of slightly overloading our previous notation, in this section $\mathcal{G} \subset \mathcal{F}$ shall denote the set of unambiguous acts.

Fixing a function $\rho : \Omega \to \{1, \ldots, N\}$, let Π_ρ denote the partition on Ω generated by ρ. That is, $\Pi_\rho = \{E_1, \ldots, E_N\}$, with $E_n = \rho^{-1}(n)$ for $n = 1, \ldots, N$. Let $\mathcal{F}_\rho$ denote the set of acts adapted to Π_ρ. That is, if $f \in \mathcal{F}_\rho$ then for all $x \in X$ and all $n \in \{1, \ldots, N\}$ either $f^{-1}(x) \cap \rho^{-1}(n) = \varnothing$ or $f^{-1}(x) \cap \rho^{-1}(n) = \rho^{-1}(n)$. For any act in $\mathcal{F}_\rho$ if we permute its outcomes on the elements of the partition Π_ρ then the resulting act is also in $\mathcal{F}_\rho$. More precisely, fix an act f in $\mathcal{F}_\rho$ and a permutation $\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}$, denote by f_σ the act in $\mathcal{F}_\rho$ for which

$$f_\sigma(\omega) = f\left(\omega'\right) \text{ where } \omega' \in \rho^{-1} \circ \sigma \circ \rho(\omega).$$

The next definition states properties of the preference relation that should hold with regard to a partition of unambiguous events that the DM assesses as all being equally likely. To see why, notice that if the function ρ generates the partition $\Pi_\rho = \{E_1, \ldots, E_N\}$, with $E_n = \rho^{-1}(n)$ for $n = 1, \ldots, N$, and the DM's prior π places equal-weight (of $1/N$) on each element of Π_ρ, then for any act f in $\mathcal{F}_\rho$ it follows that $\pi \circ f^{-1} = \pi \circ f_\sigma^{-1}$ for *every* permutation $\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}$. Hence probabilistic sophistication requires that all these acts must reside in the same indifference set.

Definition 8 *(Strongly uniform (unambiguous) partitions).* Fix $\rho : \Omega \to \{1, \ldots, N\}$. The partition Π_ρ is *strongly uniform* if

(i) $\rho^{-1}(n) \in \mathcal{U}$ for all $n = 1, \ldots, N$; and,
(ii) for every f in $\mathcal{F}_\rho$

$$f \sim f_\sigma \text{ for every permutation } \sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}.$$

The final definition states the form of convergence we require for unambiguous acts that ensures the set of unambiguous events $\mathcal{U}$ satisfies the countable closure condition $R.3$ of a λ-system.

Definition 9 *(Converging (in Preference) Sequences).* A sequence of (unambiguous) acts $\{g_n\}_{n=1}^\infty$ in $\mathcal{G}$ *converges in preference* to g_∞ if for any two (unambiguous) acts g_* and g^* in $\mathcal{G}$ satisfying $g^* \succ g_\infty \succ g_*$ there exists an integer N such that $g^* \succ g_n \succ g_*$ whenever $n \geqslant N$.

The following five postulates (with their names) are taken from Epstein and Zhang (2001), albeit with the adaptations appropriate for Gul and Pesendorfer's (2014) setting with bounded monetary outcomes.

Eventwise Monotonicity (Postulate 1) and *Weak Comparative Probability* (Postulate 2) are Savage's (1954) postulates P3 and P4, respectively, with the former being necessary (and sufficient) for the certainty equivalent representation of the restriction of the DM's preferences to unambiguous acts to exhibit stochastic monotonicity, and the latter ensuring that the derived likelihood relation over events obtained from the DM's "betting preferences" do not depend on the stakes of those bets.[9]

[9] Recall *Eventwise Monotonicity* (Savage's (1954) P3) is implied by Axiom 5.

Small Unambiguous Event Continuity (Postulate 3) ensures the set of unambiguous events is rich in the sense that any partition of *non-diffuse* events can be refined into a partition of (smaller) *unambiguous* events. As Epstein and Zhang (2001) highlight, its formulation differs from Savage's P6 so as to allow for the possibility that unambiguous events may not be closed under intersection. In conjunction with *Monotone Continuity* (Postulate 4) it also plays a crucial role in establishing the mixture monotonicity of the certainty equivalent representation. In addition, Monotone Continuity guarantees the agreeing probability over the set of unambiguous events is both countably additive and convex-ranged.

Strong Partition Neutrality (Postulate 5), the fifth and final postulate taken from Epstein and Zhang (2001), ensures for any pair of uniform partitions of unambiguous events with common cardinality, if we take one act that is adapted to one partition and another act that is adapted to the other partition, then those two acts are mapped by the agreeing probability to the *same* lottery only if they resided in the same indifference set.[10]

Postulate 1 *(Eventwise Monotonicity). For any non-null unambiguous event $R \in \mathcal{U} \setminus \mathcal{N}$, any act $f \in \mathcal{F}$ and any pair of consequences $x, y \in X$, $x > y$ if and only if $x_R f \succ y_R f$.*

Postulate 2 *(Weak Comparative Probability). For all $w, x, y, z \in X$ and all pairs of unambiguous events $R, R' \in \mathcal{U}$, if $w > z$ and $x > y$ then*

$$w_R z \succsim w_{R'} z \implies x_R y \succsim x_{R'} y\,.$$

Postulate 3 *(Small Unambiguous Event Continuity). Fix a pair of functions $\rho_1 : \Omega \to \{1, \ldots, n\}$ and $\rho_2 : \Omega \to \{1, \ldots, m\}$ such that $\rho_1^{-1}(i) \in \mathcal{U}$ for all i in $\{1, \ldots, n\}$ and $\rho_2^{-1}(j) \in \mathcal{U}$ for all j in $\{1, \ldots, m\}$. Then, for any pair of (unambiguous) acts $f \in \mathcal{F}_{\rho_1}$ and $g \in \mathcal{F}_{\rho_2}$, if $f \succ g$ then there exists two functions $\hat{\rho}_1 : \Omega \to \{1, \ldots, N\}$ and $\hat{\rho}_2 : \Omega \to \{1, \ldots, M\}$ such that $\Pi_{\hat{\rho}_1}$ refines Π_{ρ_1}, $\Pi_{\hat{\rho}_2}$ refines Π_{ρ_2}, and*

$$\hat{\rho}_1^{-1}(i) \in \mathcal{U},\ x_{\hat{\rho}_1^{-1}} f \succ g \text{ for all } i \in \{1, \ldots, N\},$$

$$\hat{\rho}_2^{-1}(j) \in \mathcal{U},\ f \succ x_{\hat{\rho}_2^{-1}} g \text{ for all } j \in \{1, \ldots, M\}.$$

Postulate 4 *(Monotone Continuity). For any unambiguous event R in $\mathcal{U}$, any pair of outcomes $x^* > x$, any unambiguous act g in $\mathcal{G}$, and decreasing sequence of unambiguous events $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{U}$ with $A_1 \subseteq R$ define*

$$R_n = R \cap (\Omega \backslash A_n),\ A_\infty = \bigcap\nolimits_{n=1}^{\infty} A_n,\ R_\infty = R \cap (\Omega \backslash A_\infty),$$

$$f_n = x^*_{A_n} x_{R_n} g,\ f_\infty = x^*_{A_\infty} x_{R_\infty} g.$$

If f_n is in $\mathcal{G}$ for all $n = 1, 2, \ldots$ then $\{f_n\}_{n=1}^{\infty}$ converges in preference to f_∞ and f_∞ is in $\mathcal{G}$.

Postulate 5 *(Strong Partition Neutrality). Fix a pair of functions $\rho_1, \rho_2 : \Omega \to \{1, \ldots, n\}$. If the partitions Π_{ρ_1} and Π_{ρ_2} are strongly uniform then for any f in $\mathcal{F}_{\rho_1}$*

$$f \sim g$$

[10] Epstein and Zhang's (2001) non-degeneracy axiom is not required since we are working with a one-dimensional outcome set with an intrinsic natural order. Hence non-degeneracy follows directly from Postulate 1.

for any act g in $\mathcal{F}_{\rho_2}$ for which there exists a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that for every $\omega \in \Omega$

$$g(\omega) = f(\omega') \text{ for some } \omega' \in \rho_1^{-1} \circ \sigma \circ \rho_2(\omega)\,.$$

Together these postulates provide a characterization of a probabilistically sophisticated preference relation in which the corresponding set of risky events is $\mathcal{U}$, the set of events satisfying Epstein and Zhang's (2001) behavioral definition for classifying an event as unambiguous.

Theorem 5. *Suppose the preference relation $\succsim$ satisfies Axiom 1 (Ordering) and Postulate 1 (Eventwise Monotonicity) and the corresponding set of unambiguous events $\mathcal{U}$ is a λ-system. Then the following are equivalent.*

1. *The preference relation $\succsim$ satisfies Postulates 2–5.*
2. *The preference relation $\succsim$ is probabilistically sophisticated with respect to $\mathcal{U}$.*

The proof of this theorem follows immediately from the proof of Epstein and Zhang (2001, Theorem 5.2, p280). However, we follow Nehring's (2006) (corrected) reformulation as we note, in contrast to Epstein and Zhang's (2001) original statement, the λ-system property of $\mathcal{U}$ is part of the hypothesis of our theorem and is not derived from the postulates.[11]

Epstein and Zhang's (2001) behavioral definition for classifying unambiguous events has not gone unchallenged. Nehring (2006) and Wakker (2008) present examples that they contend demonstrate a failure of this definition to achieve the intended separation of probabilistic risk from ambiguity. Nehring goes so far as to suggest that it may be impossible to infer what a DM views as unambiguous from a (purely) "behaviorist revealed preference approach". He suggests instead that we may need to inject more epistemic content into the definition of what is unambiguous, say by imputing probabilistic beliefs on the basis of verbal testimony of the DM or a shared understanding of certain aspects of the decision setting, an approach he develops in Nehring (2009).

It is not necessary for us to take a stand on these issues here. Rather, we conclude this section by noting that our approach is compatible with an endogenous definition of ambiguity, but does not require it. Should any approach for classifying (un)ambiguous events achieve some degree of acceptance in the literature—whether it be "behaviorist" or something with a more nuanced epistemic flavor along the lines suggested by Nerhring—it can readily be incorporated into our main representation result, Theorem 4, as the analysis in this section shows.

6. GUU in a small world

Formally a GUU certainty equivalent as specified in Definition 6 requires an infinite state space. In many applications, however, the collection of acts under consideration, even if the cardinality of this collection is uncountably infinite, has the following feature: the coarsest common refinement of the partitions of the state space associated with these acts has *finite* cardinality. That is, we can view this coarsest common refinement as corresponding to a *small world* with a finite "state" space, Θ. The space Θ is naturally interpreted as the partition of the underlying

[11] As Kopylov (2007, Theorem 5.1, p246) demonstrates, $\mathcal{U}$ need only be closed under disjoint unions of a particular kind. He refers to the resulting generalization of a λ-system, as a "mosaic".

large world (that is, infinite) state space Ω to which all the acts under consideration are adapted. The onto function $\rho\colon \Omega \to \Theta$ describes this partition in the sense that the state $\theta \in \Theta$ is the one that is realized in the small world whenever the event $\rho^{-1}(\theta)$ in the large world obtains.

For this small world, let $a\colon \Theta \to [\ell, m]$ denote a (small-world) act and let $\mathcal{A}$ denote the set of (small-world) acts. The act $a \in \mathcal{A}$ in this small world corresponds to the act $f = a \circ \rho$ in the underlying large world.

As a direct analogy to the reasoning of Gul and Pesendorfer (2014, sec. 4), in which they define an EUU maximizer for a small world, we propose the following definition.[12]

Definition 10. A preference relation $\succsim$ on $\mathcal{A}$ is a member of the set of *small world* GUU maximizers if there exists a triple $\langle \mu, u, s\rangle$, where μ is a probability on the power set of Θ, u is an unambiguous equivalent, and s is a sure equivalent, such that $\succsim$ is represented by the certainty equivalent:

$$\hat{c}(a) = s\left(\sum_{B\subseteq\Theta}[u(\min_{\theta\in B} a(\theta), \max_{\theta\in B} a(\theta))]\,\mu(B)\right).$$

One special case is where the support of the probability μ is the set of singleton events $\{\{\theta\} : \theta \in \Theta\}$ so that $\mu(A) = 0$ for all non-singleton events A. In this case, for every small-world act $a \in \mathcal{A}$, the corresponding large-world act $a \circ \rho$ is in $\mathcal{G}$ and so the certainty equivalent is given by $s\left(\sum_{\theta\in\Theta}[u(a(\theta))]\mu(\{\theta\})\right)$.

The next result, essentially a corollary of Gul and Pesendorfer's (2014) Theorem 4, establishes that if we fix a GUU maximizing preference on the large world, then each onto function $\rho\colon \Omega \to \Theta$ induces a corresponding small world GUU maximizing preference relation. Moreover, the large world prior does not constrain in any way what form the probability on the power set of Θ can take, as we can always find an onto function $\rho\colon \Omega \to \Theta$ to induce such a probability.

Theorem 6. *Fix a GUU maximizer $\succsim$ characterized by the triple $\langle \pi, u, s\rangle$ and fix a finite set Θ.*

1. *For every onto function $\rho\colon \Omega \to \Theta$, there is a unique probability μ on the power set of Θ, such that, for all $a \in \mathcal{A}$,*

$$s\left(\pi \circ u\left(\underline{a\circ\rho}, \overline{a\circ\rho}\right)^{-1}\right) = s\left(\sum_{B\subseteq\Theta}[u(\min_{\theta\in B} a(\theta), \max_{\theta\in B} a(\theta))]\,\mu(B)\right).$$

2. *For every probability μ on the power set of Θ, there exists an onto function $\rho\colon \Omega \to \Theta$, such that, for all $a \in \mathcal{A}$,*

$$s\left(\pi \circ u\left(\underline{a\circ\rho}, \overline{a\circ\rho}\right)^{-1}\right) = s\left(\sum_{B\subseteq\Theta}[u(\min_{\theta\in B} a(\theta), \max_{\theta\in B} a(\theta))]\,\mu(B)\right).$$

Echoing Gul and Pesendorfer (2014), we note that the probability μ that forms part of the characterization of a small-world GUU maximizer can be interpreted as a *basic belief assign-*

[12] As a reminder of our notation for degenerate lotteries, recall $[x]$ denotes the degenerate lottery that yields x with probability 1.

ment, a central concept in the Dempster-Shafer theory of evidence that generalizes the Bayesian theory of subjective probability to allow for imprecision in a DM's beliefs about events. From μ we can derive two capacities by setting for each $B \subseteq \Theta$: [13]

$$bel(B) := \sum_{C \subseteq B} \mu(C) \quad \text{and} \quad pl(B) := \sum_{C \cap B \neq \varnothing} \mu(C).$$

The former is called the *belief function* and is interpreted as a lower bound on the probability of an event, while the latter is called the *plausibility function* and is interpreted as an upper bound on the probability of an event. It readily follows that the two functions are conjugate duals, that is, $bel(B) = 1 - pl(\Theta \setminus B)$. The interpretation is that the quantity $bel(B)$ is the weight given to evidence that directly supports B will obtain while $pl(B)$ is the weight given to all the evidence both direct and indirect in support of B obtaining. This accords neatly with the prior on the underlying large world since it follows from the proof of Lemma 3 (the existence of an envelope for any large-world act) that

$$bel\,(B) = \sup_{R \in \mathcal{R}, R \subseteq \rho^{-1}(B)} \pi\,(R) \text{ and } pl\,(B) = \inf_{R \in \mathcal{R}, R \supseteq \rho^{-1}(B)} \pi\,(R)\;.$$

That is, noting that $\rho^{-1}(B)$ is set of large-world states that map to B, $bel(B)$ corresponds to the probability of the "largest" measurable event that lies inside $\rho^{-1}(B)$, while $pl(B)$ corresponds to the probability of the "smallest" measurable event in which B resides.

This extension to Bayesian probability theory has been incorporated into decision theory for linear (that is, expected) utility by (among others) Jaffray (1989), and so one can view Gul and Pesendorfer (2014) as providing it with a subjective foundation. Our contribution is to establish that there is nothing inherent in the Dempster-Shafer theory of evidence that necessitates the evaluation of an act via a linear expectation. That is, we have established for small worlds that it is perfectly possible to employ the Dempster-Shafer theory of evidence without requiring a Bernoulli utility.

7. Two examples

We conclude with two examples illustrating how the family of Generalized Uncertain Utility maximizers is broader than the families modeled by existing theories.

The first example shows how we can represent Expected Uncertain Utility maximizing preferences (Gul and Pesendorfer, 2014), but where the set of events the DM views as risky does not form a σ-algebra. The second example demonstrates that our theory indeed allows us to represent agents with additional combinations of risk and ambiguity preferences. Specifically, the agent's preferences over the perceived risky acts are represented by a rank-dependent utility functional. Moreover, even though the DM's prior is defined on a σ-algebra of risky events, the only events that are *ideal*, that is, satisfy the property Gul and Pesendorfer (2014) require for an event to be deemed unambiguous, are ones for which the prior assigns probability zero or probability one. Taken together, these examples show the flexibility and modularity of our approach.

[13] A capacity β is a normalized monotonic function defined over subsets of Θ. That is, $\beta(\varnothing) = 0$, $\beta(\Theta) = 1$, and $\beta(C) \geqslant \beta(B)$ whenever $B \subset C$.

Example: EUU without a σ-algebra of risky events

The state space is the unit square, that is, $\Omega = \Omega_1 \times \Omega_2 = [0,1] \times [0,1]$. Let $\mathcal{L}$ denote the set of (Lebesgue) measurable subsets of Ω.

Fix the triple $\langle \pi_1, u_1, s_1 \rangle$ where the prior π_1 will be defined in detail below, u_1 is the (extreme ambiguity-averse) unambiguous equivalent $u_1(x, y) = x$, and s_1 is the simple arithmetic mean. The preference relation $\succsim_1$ is generated by the certainty equivalent:

$$c_1(f) = s_1(\pi_1 \circ u_1^{-1}\left(\underline{f}, \bar{f}\right)) = \sum_{x \in X} \pi_1\left(\underline{f}^{-1}(x)\right) x .$$

The prior π_1 is defined by a collection of probability measures on a domain over which these probability measures agree. We take this collection of probability measures to be the set of convex combinations of a pair of probability measures μ_0 and μ_1: where μ_0 is the (unique) probability measure on $\mathcal{L}$ with support $\{(\omega_1, \omega_2) \in \Omega : \omega_1 - \omega_2 \geqslant 0\}$, and density

$$\hat{\mu}_0(\omega_1, \omega_2) = \begin{cases} 1/\omega_1 & \text{if } \omega_1 - \omega_2 \geqslant 0 \\ 0 & \text{otherwise} \end{cases} ;$$

and where μ_1 is the (unique) probability measure on $\mathcal{L}$ with support $\{(\omega_1, \omega_2) \in \Omega : \omega_1 + \omega_2 \leqslant 1\}$, and density

$$\hat{\mu}_1(\omega_1, \omega_2) = \begin{cases} 1/(1-\omega_1) & \text{if } \omega_1 + \omega_2 \leqslant 1 \\ 0 & \text{otherwise} \end{cases} .$$

For each $\alpha \in (0, 1)$, set $\mu_\alpha := \alpha\mu_1 + (1 - \alpha)\mu_0$. Hence the domain of π_1 is

$$\mathcal{R}_1 = \{E \in \mathcal{L} : \mu_\alpha(E) = \mu_1(E) \text{ for all } \alpha \in [0, 1)\} ,$$

and we set $\pi_1(E) := \mu_1(E)$ for all E in $\mathcal{R}_1$. Letting P denote the set of probability measures formed by taking convex combinations of μ_1 and μ_0, that is,

$$P = \{\mu : \mu = \alpha\mu_1 + (1 - \alpha)\mu_0 \text{ for some } \alpha \in [0, 1]\} ,$$

it readily follows that $\succsim_1$ admits the following multiple prior representation:

$$c_1(f) = \min_{\mu \in P} \sum_{x \in X} \mu\left(f^{-1}(x)\right) x .$$

Observation 1. All vertical rectangles of the form $[a, b] \times [0, 1]$ $(a < b)$ as well as all horizontal rectangles of the form $[1, 0] \times [c, d]$ $(c < d)$ are in $\mathcal{R}_1$. However, the rectangle $[a, b] \times [c, d]$ corresponding to the intersection of the pair of rectangles $[a, b] \times [0, 1]$ and $[1, 0] \times [c, d]$ is in $\mathcal{R}_1$ *only when* $a + b = 1$.

To verify these observations, first notice that since $\mu_\alpha = \alpha\mu_1 + (1 - \alpha)\mu_0$ it suffices to show $\mu_1(E) = \mu_0(E)$ to establish the event E is in $\mathcal{R}_1$. And so, since

$$\begin{aligned} \mu_1([a, b] \times [0, 1]) &= \int_a^b \left(\int_0^{1-\omega_1} \frac{1}{1-\omega_1} d\omega_2\right) d\omega_1 \\ &= \int_a^b \left[\frac{\omega_2}{1-\omega_1}\right]_{\omega_2=0}^{\omega_2=1-\omega_1} d\omega_2 = \int_a^b 1 \, d\omega_2 = b - a, \end{aligned}$$

and

$$\mu_0([a,b]\times[0,1]) = \int_a^b \left(\int_0^{\omega_1} \frac{1}{\omega_1} d\omega_2\right) d\omega_1$$
$$= \int_a^b \left[\frac{\omega_2}{\omega_1}\right]_{\omega_2=0}^{\omega_2=\omega_1} d\omega_2 = \int_a^b 1\, d\omega_2 = b-a,$$

it follows that $[a,b]\times[0,1]$ is in $\mathcal{R}_1$.[14] Furthermore, since

$$\mu_1([0,1]\times[c,d]) = \int_c^d \left(\int_0^{1-\omega_2} \frac{1}{1-\omega_1} d\omega_1\right) d\omega_2$$
$$= \int_c^d \left([-\ln(1-\omega_1)]_{\omega_1=0}^{\omega_1=1-\omega_2}\right) d\omega_2 = \int_c^d \ln\left(\frac{1}{\omega_2}\right) d\omega_2$$

and

$$\mu_0([0,1]\times[c,d]) = \int_c^d \left(\int_{\omega_2}^{1} \frac{1}{\omega_1} d\omega_1\right) d\omega_2$$
$$= \int_c^d \left([\ln(\omega_1)]_{\omega_1=\omega_2}^{\omega_1=1}\right) d\omega_2 = \int_c^d \ln\left(\frac{1}{\omega_2}\right) d\omega_2,$$

it follows that $[0,1]\times[c,d]$ is in $\mathcal{R}_1$.

Finally, the rectangle $[a,b]\times[c,d]$ is in $\mathcal{R}_1$ if and only if

$$\mu_1([a,b]\times[c,d]) = \int_a^b \int_c^d \hat{\mu}_1(\omega_1,\omega_2)\, d\omega_2 d\omega_1$$
$$= \int_a^b \int_c^d \hat{\mu}_0(\omega_1,\omega_2)\, d\omega_2 d\omega_1 = \mu_0([a,b]\times[c,d])$$

However, notice that neither $\hat{\mu}_1(\omega_1,\omega_2)$ nor $\hat{\mu}_0(\omega_1,\omega_2)$ depend on ω_2 and since $\hat{\mu}_1(\omega_1,\omega_2) = \hat{\mu}_0(1-\omega_1,\omega_2)$ it readily follows from the geometry of the intersection of the rectangle with the respective supports of μ_1 and μ_0 that the required equality of the two double-integrals holds if and only if the rectangle exhibits reflection symmetry about the line $\omega_1 = 1/2$. This in turn requires $1/2 - a = b - 1/2$.

By construction $\mathcal{R}_1$ satisfies the three properties $R.1$, $R.2$, and $R.3$ that characterize a λ-system. But as it is not closed under intersection it is not a σ-algebra.[15]

However, $\succsim_1$ does *not* conform to the Expected Uncertain Utility model of Gul and Pesendorfer (2014). To see why, we first note that every event $E \in \mathcal{R}_1$ is ideal. Formally, an event A is ideal for $\succsim_1$ if for any four acts $f, g, h, h' \in \mathcal{F}$:

$$\left[f_A h \succsim_1 g_A h \text{ and } h_A f \succsim_1 h_A g\right] \Longrightarrow \left[f_A h' \succsim_1 g_A h' \text{ and } h'_A f \succsim_1 h'_A g\right]$$

However, we can show any event in $\mathcal{R}_1$ actually satisfies a stronger property, namely Savage's sure-thing principle: for any two acts $f, g \in \mathcal{F}$ $f \succsim_1 g_A f$ implies $f_A g \succsim_1 g$. To see this, fix $E \in \mathcal{R}_1$. For any pair of acts f, g in $\mathcal{F}$:

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14 Indeed, we have established that the marginal of μ_α on Ω_1 is the uniform distribution.

15 For example, we can immediately see that the square $[1/4, 1/3]\times[1/3, 1/2]$ (formed by taking the intersection of the two rectangles $[1/4, 1/3]\times[0, 1]$ and $[0, 1]\times[1/3, 1/2]$ that are both members of $\mathcal{R}_1$) is not in $\mathcal{R}_1$ since it lies (wholly) inside the support of μ_1 and (wholly) outside the support of μ_0.

$$
\begin{aligned}
& f \succsim_1 g_E f \\
& \Leftrightarrow \min_{\mu \in P}\left(\sum_{x \in X}\left[\pi_1(E) \frac{\mu\left(f^{-1}(x) \cap E\right)}{\pi_1(E)}+\left[1-\pi_1(E)\right] \frac{\mu\left(f^{-1}(x) \cap \Omega \backslash E\right)}{\left[1-\pi_1(E)\right]}\right] x\right) \\
& \quad \geqslant \min_{\mu \in P}\left(\sum_{x \in X}\left[\pi_1(E) \frac{\mu\left(g^{-1}(x) \cap E\right)}{\pi_1(E)}+\left[1-\pi_1(E)\right] \frac{\mu\left(f^{-1}(x) \cap \Omega \backslash E\right)}{\left[1-\pi_1(E)\right]}\right] x\right) \\
& \Leftrightarrow \pi_1(E) \min_{\mu \in P}\left(\sum_{x \in X} \frac{\left[\mu\left(f^{-1}(x) \cap E\right)\right]}{\pi_1(E)} x\right) \\
& \quad +\left[1-\pi_1(E)\right] \min_{\mu \in P}\left(\sum_{x \in X} \frac{\left[\mu\left(f^{-1}(x) \cap \Omega \backslash E\right)\right]}{\pi_1(E)} x\right) \\
& \quad \geqslant \pi_1(E) \min_{\mu \in P}\left(\sum_{x \in X} \frac{\left[\mu\left(g^{-1}(x) \cap E\right)\right]}{\pi_1(E)} x\right) \\
& \quad +\left[1-\pi_1(E)\right] \min_{\mu \in P}\left(\sum_{x \in X} \frac{\left[\mu\left(f^{-1}(x) \cap \Omega \backslash E\right)\right]}{\pi_1(E)} x\right) \\
& \Leftrightarrow \pi_1(E) \min_{\mu \in P}\left(\sum_{x \in X} \frac{\left[\mu\left(f^{-1}(x) \cap E\right)\right]}{\pi_1(E)} x\right) \\
& \quad +\left[1-\pi_1(E)\right] \min_{\mu \in P}\left(\sum_{x \in X} \frac{\left[\mu\left(g^{-1}(x) \cap \Omega \backslash E\right)\right]}{\pi_1(E)} x\right) \\
& \quad \geqslant \pi_1(E) \min_{\mu \in P}\left(\sum_{x \in X} \frac{\left[\mu\left(g^{-1}(x) \cap E\right)\right]}{\pi_1(E)} x\right) \\
& \quad +\left[1-\pi_1(E)\right] \min_{\mu \in P}\left(\sum_{x \in X} \frac{\left[\mu\left(g^{-1}(x) \cap \Omega \backslash E\right)\right]}{\pi_1(E)} x\right) \\
& \Leftrightarrow f_E g \succsim_1 g
\end{aligned}
$$

Hence, $f \succsim_1 g_E f \Leftrightarrow f_E g \succsim_1 g$. But as we have just shown above, $\mathcal{R}_1$ is not a σ-algebra. So for these preferences, the set of ideal events is not a σ-algebra, in particular, the set of ideal events is not closed under intersection.

Example: rank-dependent risk-preferences

The state space is the unit interval. Set $\mathcal{R}_2$ to be the set of (Lebesgue) measurable subsets of $\Omega = [0, 1]$.

Fix the triple $\langle \pi_2, u_2, s_2 \rangle$ We take the prior π_2 to be the Lebesgue measure and u_2 to be the (extreme ambiguity-averse) unambiguous equivalent $u_2(x, y) = x$. The sure equivalent, s_2, is from the family of rank-dependent (generalized) means and defined as

$$
s_2(L) = \sum_{x \in X}\left[\left(\sum_{y \geqslant x} L(y)\right)^2 - \left(\sum_{z > x} L(z)\right)^2\right] x .
$$

The preference relation $\succsim_2$ is generated by the certainty equivalent:

$$c_2(f) = s_2\left(\pi_2^{-1} \circ \underline{f}\right).$$

By construction, $\mathcal{R}_2$, the domain of the prior π_2, is a σ-algebra. However, the preference relation $\succsim_2$, generated by the functional $c_2(\cdot)$, is not from the class of EUU preferences. This is because not all events in the domain of the prior are *ideal* in the sense of Gul and Pesendorfer (2014). For an event $E \subset \Omega$ to be *ideal* requires for any four acts f, g, h and h' in $\mathcal{F}$,

$$\left[f_E h \succsim g_E h \text{ and } h_E f \succsim h_E g\right] \Longrightarrow \left[f_E h \succsim g_E h \text{ and } h_E f \succsim h_E g\right].$$

It turns out that the only events that satisfy this property are ones for which the prior assigns probability zero (that is, the event is null) or probability one (that is, the event is universal).

Observation 2. Fix an event E in $\mathcal{R}_2$. If $\pi_2(E) \in (0, 1)$ then E is *not* ideal.

To show this, first fix a probability $p \in (0, 1)$ and consider the following four lotteries, L_1, L_2, L_3, L_4 given by

$$L_1(x) = \begin{cases} p/2 & \text{if } x \in \{0, 1\} \\ 1-p & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}, \quad L_2(x) = [y]$$

$$L_3(x) = \begin{cases} p/2 & \text{if } x = 1 \\ 1-p/2 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}, \quad L_4(x) = \begin{cases} p & \text{if } x = y \\ 1-p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases},$$

where $y \in (p/4, 1/4)$.

The rank-dependent mean of the four lotteries are:

$$s_2(L_1) = \left(\frac{p}{2}\right)^2 + \left[\left(1 - \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2\right] y = \frac{p^2}{4} + (1-p)\, y$$

$$s_2(L_2) = y, \; s_2(L_3) = \frac{p^2}{4} \text{ and } s_2(L_4) = p^2 y$$

Hence,

$$s_2(L_1) - s_2(L_2) = \frac{p^2}{4} - py < 0, \text{ as } y > \frac{p}{4},$$

$$\text{and } s_2(L_3) - s_2(L_4) = \frac{p^2}{4} - p^2 y > 0, \text{ as } y < \frac{1}{4}.$$

Now consider any event E in $\mathcal{R}_2$ for which $\pi_2(E) = p$. Since π is convex-ranged, there exists a subset of E, say A, such that $\pi_2(A) = \pi_2(E)/2$ and a subset of $\Omega \backslash E$, say B, such that $\pi_2(B) = (1 - \pi_2(E))/2$. Consider the following four acts $f, g, h, h' \in \mathcal{G}$, where $f = 1_{A \cup B} 0$, $h' = 0$ and $g = h = y$, for some $y \in (\max\{\pi_2(E), 1 - \pi_2(E)\}/4, 1/4)$.

Hence we have:

$$f_E h = 1_A 0_{(\Omega \backslash A) \cap E}\, y, \; g_E h = y, \; h_E f = 1_B 0_{(\Omega \backslash B) \cap (\Omega \backslash E)}\, y, \; h_E g = y,$$

$$f_E h' = 1_A 0, \; g_E h' = y_E 0, \; h'_E f = 1_B 0, \; h'_E g = 0_E y.$$

Since all eight acts above are measurable with respect to $\mathcal{R}_2$ we can map each of them to a lottery in $\Delta_0(X)$. And we have:

$$\pi_2 \circ (f_E h)^{-1}(x) = \begin{cases} \pi_2(E)/2 & \text{if } x \in \{0,1\} \\ 1-\pi_2(E) & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}, \pi_2 \circ (g_E h)^{-1} = [y]$$

$$\pi_2 \circ (h_E f)^{-1}(x) = \begin{cases} [1-\pi_2(E)]/2 & \text{if } x \in \{0,1\} \\ \pi_2(E) & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}, \pi_2 \circ (h_E g)^{-1} = [y],$$

$$\pi_2 \circ \left(f_E h'\right)^{-1}(x) = \begin{cases} \pi_2(E)/2 & \text{if } x = 1 \\ 1-\pi_2(E)/2 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases},$$

$$\pi_2 \circ \left(g_E h'\right)^{-1}(x) = \begin{cases} \pi_2(E) & \text{if } x = y \\ 1-\pi_2(E) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases},$$

$$\pi_2 \circ \left(h'_E f\right)^{-1}(x) = \begin{cases} [1-\pi_2(E)]/2 & \text{if } x = 1 \\ 1/2+\pi_2(E)/2 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases},$$

$$\pi_2 \circ \left(h'_E g\right)^{-1}(x) = \begin{cases} 1-\pi_2(E) & \text{if } x = y \\ \pi_2(E) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Applying $s_2(\cdot)$ to obtain the rank-dependent means of these eight lotteries, it follows from the calculations we performed above that:

$$f_E h \succ_2 g_E h \text{ and } h_E f \succ_2 h_E g,$$

but $g_E h' \succ_2 f_E h'$ (as well as $h'_E g \succ_2 h'_E f$).

Hence the event E is not ideal. As p was an arbitrarily fixed element from the open interval (0, 1), this establishes the validity of the observation.

Appendix

Proof of Lemma 2. Fix an arbitrary lottery $L \in \Delta_0(X)$ with finite support $\{x_1, \ldots, x_n\}$. For each $i = 1, \ldots, n$, set $p^i := \sum_{j=1}^{i} L(x_j)$. Set $R^n := \Omega$ (hence, $\pi(R^n) = 1 = p^n$). Since the prior π is convex-valued and $\mathcal{R}$ is a λ-system, it follows that there exists a risky event $R^{n-1} \in \mathcal{R}$, such that $\pi(R^{n-1}) = p^{n-1}$. Furthermore, there exists a risky event $R^{n-2} \in \mathcal{R}$ for which $R^{n-2} \subset R^{n-1}$, and $\pi(R^{n-2}) = p^{n-2}$. Repeating this process for $j = n-3$ down to $j = 1$, we obtain a sequence of increasing risky events $R^1 \subset R^2 \subset \ldots \subset R^{n-1} \subset R^n = \Omega$. Applying condition *R.4* of a λ-system (established by Lemma 1) it follows that for each $i = 2, \ldots, n$, the event $R^i \backslash R^{i-1}$ is also in $\mathcal{R}$ with $\pi(R^i \backslash R^{i-1}) = p^i - p^{i-1} = L(x_i)$. By construction the collection of risky events $\left\{R^1, R^2 \backslash R^1, \ldots, R^{n-1} \backslash R^{n-2}, \Omega \backslash R^{n-1}\right\}$ forms a partition of Ω. So take f to be the act in which $f(\omega) = x_1$ if $\omega \in R^1$ and $f(\omega) = x_i$ if $\omega \in R^i \backslash R^{i-1}$ for some $i = 2, \ldots n$. By construction $\pi \circ f^{-1} = L$, as required. □

Proof of Lemma 3. In order to construct a risky envelope for an arbitrary act it is convenient to define a capacity associated with the prior π, its inner measure, denoted by $\underline{\pi}$.[16] For an arbitrary event $A \subseteq \Omega$, set

$$\underline{\pi}(A) := \sup_{R \in \mathcal{R}, R \subseteq A} \pi(R),$$

and define $\underline{A} \in \mathcal{R}$ to be its *inner sleeve*, where $\underline{A} \subseteq A$ for which $\pi(\underline{A}) = \underline{\pi}(A)$. Such a π-inner sleeve exists since π is convex-ranged.

Fix an arbitrary act f, with range $f(\Omega) = \{x_1, \ldots, x_n\}$. Let $N = \{1, \ldots, n\}$ and set $A_i := f^{-1}(x_i)$, so that $\{A_i\}_{i \in N}$ forms a finite partition of Ω. Let $\mathcal{P}$ denote the set of all non-empty subsets of N and for each $J \in \mathcal{P}$, set $\mathcal{P}^J = \{L \in \mathcal{P} \mid L \subseteq J\}$. Set $A^J = \bigcup_{i \in J} A_i$ and let $\underline{A}^J$ be its inner sleeve. Following the approach of Gul and Pesendorfer (2014, pp21-2) in their Appendix A, we construct the partition $\left\{R^J\right\}_{J \in \mathcal{P}}$ of Ω, that we dub a *risky split* of the partition $\{A_i\}_{i \in N}$ inductively as follows:

$$R^{\{i\}} = \underline{A_i}, \text{ for all } i \in N;$$

and, for every J such that $|J| > 1$,

$$R^J := \underline{A}^J \setminus \left(\bigcup_{L \in \mathcal{P}^J, L \neq J} R^L \right).$$

Notice that from property *R.4* of λ-systems, it follows that $R^J \in \mathcal{R}$, for every $J \in \mathcal{P}$.

Let $\underline{f}$ (respectively, $\bar{f}$) be the risky act in $\mathcal{G}$, given by $\underline{f}(\omega) = \min_{i \in J} x_i$ (respectively, $\bar{f}(\omega) = \max_{i \in J} x_i$) for $\omega \in R^J$. The fact that $\left(\underline{f}, \bar{f}\right)$ constitutes a risky envelope of f follows from Lemma A1 of Gul and Pesendorfer (2014, p22). □

Proof of Theorem 4. *Necessity*

Since the prior π is convex-valued and defined on a domain that is a λ-system, by construction the functional $c(\cdot)$ generates a preference relation that is complete, transitive, monotonic and continuous, hence Axioms 1–3 are necessary.

To establish the necessity of Axiom 4, fix a risky event $R \in \mathcal{R}$ for which $\pi(R) > 0$ and two diffuse events $D, D' \in \mathcal{D}$. Consider the pair of acts $f = y_{D \cap R} x$ and $f' = y_{R \cap D'} x$. If $y \leqslant x$ then $\underline{f} = y_R x = \underline{f}'$ and $\bar{f} = x = \bar{f}'$. Alternatively, if $y > x$ then $\underline{f} = x = \underline{f}'$ and $\bar{f} = y_R x = \bar{f}'$. Hence for all y and x it follows from the definitions of c that $c(y_{D \cap R} x) = c(y_{D' \cap R} x)$. Hence we require $y_{D \cap R} x \sim y_{D' \cap R} x$ which Axiom 4 delivers.

Finally, to see that Axiom 5 holds for a preference relation generated by a GUU certainty equivalent $c(\cdot)$ fix a risky event $R \in \mathcal{R}$ and a pair of diffuse events $D, \widehat{D} \in \mathcal{D}$. For any four prizes x, x^*, y, y^*, without loss of generality, suppose $x^* \geqslant x$ and $y^* \geqslant y$. Then

$$c\left(x^*_D x_{\Omega \setminus D}\right) = u\left(x, x^*\right) \text{ and } c\left(y^*_{\widehat{D}} y_{\Omega \setminus \widehat{D}}\right) = u\left(y, y^*\right),$$

$$c\left(x^*_{D \cap R} x_{R \setminus D} f\right) = s\left(\pi(R)\left[u\left(x, x^*\right)\right] + [1 - \pi(R)] L_{f_{\Omega \setminus R}}\right),$$

$$c\left(y^*_{\widehat{D} \cap R} y_{R \setminus \widehat{D}} f\right) = s\left(\pi(R)\left[u\left(y, y^*\right)\right] + [1 - \pi(R)] L_{f_{\Omega \setminus R}}\right)$$

[16] A capacity ν is a normalized monotonic function defined over subsets of Ω. That is, $\nu(\varnothing) = 0$, $\nu(\Omega) = 1$, and $\nu(E) \geqslant \nu(A)$ whenever $A \subset E$.

where $L_{f_{\Omega\backslash R}}$ is the lottery given by

$$L_{f_{\Omega\backslash R}}(z)=\frac{\pi\left(\left\{\omega\in\Omega:u\left(\underline{f}(\omega),\bar{f}(\omega)\right)=z\right\}\right)}{[1-\pi(R)]}.$$

Since s is monotonic, it follows that

$$s\left(\pi(R)[u\left(x,x^*\right)]+[1-\pi(R)]\,\ell_{f_{\Omega\backslash R}}\right)\geqslant s\left(\pi(R)[u\left(y,y^*\right)]+[1-\pi(R)]\,\ell_{f_{\Omega\backslash R}}\right)$$

if and only if $u(x,x^*)\geqslant u(y,y^*)$. Hence, $x^*_D x_{\Omega\backslash D}\succsim y^*_{\widehat{D}}y_{\Omega\backslash\widehat{D}}$ if and only if $x^*_{D\cap R}x_{R\backslash D}f\succsim y^*_{\widehat{D}\cap R}y_{R\backslash\widehat{D}}f$, as indeed Axiom 5 delivers. □

Sufficiency

We know from Gul and Pesendorfer (2014, Lemma A2, p22) that the set of diffuse events $\mathcal{D}$ is non-empty. So we set the ambiguity equivalent $u(x,y):=z$, where we know from the application of Axioms 2 and 3, z is the unique outcome for which $y_D x\sim z$.

The remainder of the proof follows closely the outline of the proof of Theorem 1 in Gul and Pesendorfer (2014). Fix an act f and let $\{x_1,\ldots,x_N\}=f(\Omega)$, that is the range of f. Consider the partition

$$\left\{A_n:A_n=f^{-1}(x_n)\;n=1,\ldots,N\right\}.$$

It also follows from Lemma A2 of Gul and Pesendorfer (2014) that we can partition Ω into any finite number of diffuse events $D_1,\ldots,D_L$. Since π is convex-ranged, given any $\alpha_1,\ldots,\alpha_R>0$ such that $\sum_k\alpha_k=1$, we can also construct a partition of risky events $R_1,\ldots,R_K$ such that $\alpha_k=\pi(R_k)$ for all k. Moreover, this partition can be constructed so that for all $(k,\ell)\in\{1,\ldots,K\}\times\{1,\ldots,L\}$, there is some $n\in\{1,\ldots,N\}$ with $R_k\cap D_\ell\subset A_n$.

For each risky event R_k, set:

$$\bar{x}_k:=\max_{\omega\in R_k}f(\omega)\quad\text{and}\quad\underline{x}_k=\min_{\omega\in R_k}f(\omega)$$

Define f_- and f_+ as follows:

$$f_-(\omega):=\begin{cases}\underline{x}_k & \text{if }\omega\in R_k\backslash f^{-1}(\bar{x}_k)\\ \bar{x}_k & \text{if }\omega\in f^{-1}(\bar{x}_k)\cap R_k\end{cases}$$

$$f_+(\omega):=\begin{cases}\underline{x}_k & \text{if }\omega\in f^{-1}\left(\underline{x}_k\right)\cap R_k\\ \bar{x}_k & \text{if }\omega\in R_k\backslash f^{-1}\left(\underline{x}_k\right)\end{cases}$$

By construction $f_-\leqslant f\leqslant f_+$ and $\left[\underline{f}_-,\bar{f}_-\right]=\left[\underline{f},\bar{f}\right]=\left[\underline{f}_+,\bar{f}_+\right]$ Hence by Axioms 2 and 4 $f_-\sim f_+$ ($\sim f$). Now consider the risky act g defined by setting $g(\omega)=z_k=u\left(\underline{x}_k,\bar{x}_k\right)$ whenever $\omega\in R_k$. It follows from Axiom 5 that $g\sim f$. Since g is by construction an unambiguous act, we have $x^*\sim g\sim f$ for some x^* such that

$$x^*=s(\pi\circ g^{-1})=s\left(\sum_{k=1}^K\pi(R_k)[z_k]\right)=s\left(\sum_{k=1}^K\pi(R_k)[u(\underline{x}_k,\bar{x}_k)]\right)=c(f_+)=c(f).$$

Thus we have completed the characterization of a GUU certainty equivalent $c(\cdot)$ that represents a preference relation $\succsim$ satisfying Axioms 1–5. □

Proof of Theorem 6. Fix $\langle \pi, u, s\rangle$ and the set $\Theta = \{1, \ldots, n\}$.

Part 1.

Take an arbitrary onto function $\rho : \Omega \to S$. Let $\left\{R^B\right\}_{B\subseteq\Theta}$ be the risky split of the partition $\left\{\rho^{-1}(\theta)\right\}_{\theta\in\Theta}$ as defined in the proof of Lemma 3. Fix an arbitrary act $a \in \mathcal{A}$. Let $f \in \mathcal{F}$ be the large world act $f = a \circ \rho$. For any $\omega \in R^B$, it follows from the proof of Lemma 3 that for the risky envelope $\left(\underline{f}, \bar{f}\right)$ we have

$$\underline{f}(\omega) = \min_{\hat{\omega}\in R^B} f\left(\hat{\omega}\right) = \min_{\theta\in B} a(\theta) \text{ and } \bar{f}(\omega) = \max_{\hat{\omega}\in R^B} f\left(\hat{\omega}\right) = \max_{\theta\in B} a(\theta).$$

Hence we have

$$s\left(\pi \circ u\left(\underline{f}, \bar{f}\right)^{-1}\right) = s\left(\sum_{B\subseteq\Theta}\left[u(\min_{\theta\in B} a(s), \max_{\theta\in B} a(s))\right]\mu(B)\right),$$

as required. □

Part 2.

Fix μ, an arbitrary probability distribution over the set of subsets of Θ. To construct the onto function $\rho_\mu : \Omega \to \Theta$, first choose a partition $\left\{R^B \in \mathcal{R}\colon B \subseteq S\right\}$ of Ω into risky events satisfying $\pi(R^B) = \mu(B)$, for all $B \subseteq \Theta$, $B \neq \varnothing$. For each B such that $|B| > 1$, let $\left\{D^B_\theta \in \mathcal{D}\colon \theta \in B\right\}$ be a partition of Ω into $|B|$ diffuse events. Define ρ_Θ as follows: if $\omega \in E^{\{\theta\}}$ for some $\theta \in \Theta$ then set $\rho_\mu(\omega) = \theta$, otherwise if $\omega \in R^B \cap D^B_\theta$ for some B such that $|B| > 1$ and for some $\theta \in \Theta$ then assign $\rho_\mu(\omega) = \theta$. This construction entails that $\left\{R^B \mid B \subseteq \Theta\right\}$ is a risky split of $\left\{\rho_\mu^{-1}(\theta)\right\}_{\theta\in\Theta}$. This in turn means that for any small world act $a \in \mathcal{A}$, for the risky envelope of the corresponding large world act $f = a \circ \rho_\mu$, we have

$$\underline{f}(\omega) = \min_{\theta\in B} a(\theta) \text{ and } \bar{f}(\omega) = \max_{\theta\in B} a(\theta) \text{ for } \omega \in R^B.$$

Hence,

$$\hat{c}(a) = c(f) = s\left(\pi \circ u\left(\underline{f}, \bar{f}\right)^{-1}\right) = s\left(\sum_{B\subseteq\Theta}\left[u(\min_{\theta\in B} a(\theta), \max_{\theta\in B} a(\theta))\right]\mu(B)\right),$$

as required. □

References

Abdellaoui, M., Wakker, P.P., 2005. The likelihood method for decision under uncertainty. Theory Decis. 58 (1), 3–76.

Baillon, A., L'Haridon, O., Placido, L., 2011. Ambiguity models and the Machina paradoxes. Am. Econ. Rev. 101 (4), 1547–1560.

Banach, S., Tarski, A., 1924. Sur la décomposition des ensembles de points en parties respectivement congruentes. Fundam. Math. 6 (1), 244–277.

Binmore, K.G., 2009. Rational Decisions. Princeton University Press, Princeton, NJ.

Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., Montrucchio, L., 2012. Probabilistic sophistication, second order stochastic dominance and uncertainty aversion. J. Math. Econ. 48 (5), 271–283.

Chew, S.H., Sagi, J.S., 2008. Small worlds: modeling attitudes toward sources of uncertainty. J. Econ. Theory 139 (1), 1–24.

Cohen, M., 1992. Security level, potential level, expected utility theory: a three-criteria decision model under risk. Theory Decis. 33 (2), 101–134.

Cohen, M., Jaffray, J.-Y., Said, T., 1987. Experimental comparison of individual behavior under risk and under uncertainty for gains and for losses. Organ. Behav. Hum. Decis. Process. 39 (1), 1–22.
Dean, M., Ortoleva, P., 2017. Allais, Ellsberg, and preferences for hedging. Theor. Econ. 12 (1), 377–424.
Epstein, L.G., Zhang, J., 2001. Subjective probabilities on subjectively unambiguous events. Econometrica 69 (2), 265–306.
Gilboa, I., 1988. A combination of expected utility and maxmin decision criteria. J. Math. Psychol. 32 (4), 405–420.
Grant, S., Rich, P., Stecher, J., 2019. An ordinal theory of worst- and best-case expected utility. Working paper. Australian National University, Hamburg University, University of Bayreuth, and University of Alberta.
Gul, F., 1991. A theory of disappointment aversion. Econometrica 59 (3), 667–686.
Gul, F., Pesendorfer, W., 2014. Expected uncertain utility. Econometrica 82 (1), 1–39.
Heal, G., Millner, A., 2014. Uncertainty and decision making in climate change economics. Rev. Environ. Econ. Policy 8 (1), 120–137.
Hill, B., 2019. Beyond uncertainty aversion. Research paper, HEC Paris.
Hurwicz, L., 1951. Optimality criteria for decision making under ignorance. Discussion Paper, Statistics 370, Cowles Commission.
Jaffray, J.-Y., 1988. Choice under risk and the security factor: an axiomatic model. Theory Decis. 24 (2), 169–200.
Jaffray, J.-Y., 1989. Linear utility theory for belief functions. Oper. Res. Lett. 8 (2), 107–112.
Jevons, W.S., 1877. The Principles of Science: A Treatise on Logic and Scientific Method. Classworks.
Keynes, J.M., 1921. A Treatise on Probability, Reprinted 2008 by Rough Draft Printers.
Kopylov, I., 2007. Subjective probabilities on "small" domains. J. Econ. Theory 133 (1), 236–265.
L'Haridon, O., Placido, L., 2010. Betting on Machina's reflection example: an experiment on ambiguity. Theory Decis. 69 (3), 375–393.
Lleras, J.S., Piermont, E., Svoboda, R., 2019. Asymmetric gain-loss reference dependence and attitudes toward uncertainty. Econ. Theory 68 (3), 669–699.
Maccheroni, F., Marinacci, M., Rustichini, A., 2006. Ambiguity aversion, robustness, and the variational representation of preferences. Econometrica 74 (6), 1447–1498.
Machina, M.J., 2009. Risk, ambiguity, and the rank-dependence axioms. Am. Econ. Rev. 99 (1), 385–392.
Machina, M.J., Schmeidler, D., 1992. A more robust definition of subjective probability. Econometrica 60 (4), 745–780.
Machina, M.J., Schmeidler, D., 1995. Bayes without Bernoulli: simple conditions for probabilistically sophisticated choice. J. Econ. Theory 67 (1), 106–128.
Marinacci, M., 2002. Probabilistic sophistication and multiple priors. Econometrica 70 (2), 755–764.
Nakamura, Y., 1995. Probabilistically sophisticated rank dependent utility. Econ. Lett. 48 (3–4), 441–447.
Nehring, K., 2006. Is It Possible to Define Subjective Probabilities in Purely Behavioral Terms?
Nehring, K., 2007. Bernoulli without Bayes: a theory of utility-sophisticated preferences under ambiguity. Working paper. University of California, Davis.
Nehring, K., 2009. Imprecise probabilistic beliefs as a context for decision-making under ambiguity. J. Econ. Theory 144 (3), 1054–1091.
Oxtoby, J.C., 1980. Measure and Category: A Survey of the Analogies Between Topological and Measure Spaces, vol. 2, 2nd edition. Springer-Verlag, New York.
Raiffa, H., 1961. Risk, ambiguity, and the Savage axioms: a comment. Q. J. Econ. 75 (4), 690–694.
Savage, L.J., 1954. The Foundations of Statistics. John Wiley & Sons.
Siniscalchi, M., 2009. Vector expected utility and attitudes toward variation. Econometrica 77 (3), 801–855.
Strzalecki, T., 2011. Probabilistic sophistication and variational preferences. J. Econ. Theory 146 (5), 2117–2125.
Wakker, P., 2008. New Palgrave Dictionary of Economics, 2nd ed. Palgrave Macmillan Basingstoke, pp. 428–439. Chapter Uncertainty.