Characterizing Expected Uncertain Utility

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1 Introduction

In an innovative paper, replete with many important results and insights, Gul and Pesendorfer (2014) (hereafter GP) propose a novel model for choice under uncertainty. They consider a setting of purely subjective uncertainty in which the objects of choice are acts that, for each state of nature $\omega \in \Omega$, deliver a monetary prize x from a set of final prizes $X = [\ell, m]$, with $\ell < m$. We denote the set of acts by $\mathcal{F}$, and the decision-maker's preference relation defined over $\mathcal{F}$ by $\succsim$.

In GP's model the decision-maker (hereafter, DM) has a prior μ defined over $\mathcal{E}$, a σ-algebra of what they refer to as *ideal events*. GP interpret any ideal event E (in $\mathcal{E}$) as one for which the DM can precisely quantify that event's uncertainty by assigning it the probability $\mu(E)$. An event is deemed ideal by the DM if both it and its complement together satisfy a version of Savage's (1954) sure thing principle.

The utility of any act g that is adapted to the set of ideal events (what GP refer to as an *ideal* act) may be expressed as an expected utility:

$$V(g) = \int v(g)\, d\mu \tag{1}$$

for some Bernoulli utility v.

In evaluating any general act f in their model, the DM first forms an ideal (greatest) lower bound $[f]_1$ and an ideal (least) upper bound $[f]_2$ to represent the range of possible outcomes implied by the uncertainty that she cannot precisely quantify with her prior μ. The expected utility of f with which the DM will compare the desirability of f compared to other acts is then given by:

$$V(f) = \int u([f]_1, [f]_2)\, d\mu\,, \tag{2}$$

where $u(x,y)$ (with $x \leqslant y$) is the utility assigned by the DM to an unquantifiable uncertain prospect with prizes lying in the interval $[x,y]$. GP refer to such a DM as an *expected uncertain utility (EUU)*-maximizer and to the utility index u as an *interval utility*. As they note, when f is ideal its lower and upper bounds coincide and so expression (2) reduces to the expected utility formula in (1) for the Bernoulli utility $v(x) := u(x,x)$.

In order for a preference relation to admit an expected uncertain utility representation of the form given in (2), we require it exhibit properties that ensure the existence of a rich σ-algebra of ideal events that enable us to associate with each act f its *envelope*, formally the mapping

$$[f]\colon \{[x,y] \in X \times X\colon x \leqslant y\} \to \mathbb{R}\,,$$

defined by setting for each ω in Ω, $[f](\omega) := [[f]_1(\omega), [f]_2(\omega)]$. Furthermore, any pair of acts associated with the same envelope must reside in the same indifference class. This in turn allows us to derive from the original preference relation an induced preference relation over envelopes. The characterization of EUU maximization then boils down to establishing this induced relation over envelopes admits an SEU representation characterized by a prior μ defined over the set of ideal events and an interval utility $u(\cdot,\cdot)$.

Unfortunately, GP's characterization fails on two accounts as their axioms neither ensure

(i) the set of ideal events is countably additive;

nor,

(ii) the interval utility is state-independent.

In this note we show a characterization of EUU-maximization can be obtained by means of a slight modification to GP's continuity axiom and the addition of one natural axiom that guarantees the state-independence of the interval utility. But first, we present in section 2 an example of an EUU-functional involving a state-dependent interval utility and show that the preferences generated by this example, despite satisfying all of GP's axioms, cannot be represented by an EUU function of the form in (2).

2 An example with a state-dependent interval utility.

Let the state space $\Omega = [0,1]$ be endowed with the Lebesgue measure μ. Let $\mathcal{E}_\mu$ denote the set of measurable events with respect to μ. Following GP, $[f]$ is the (interval) envelope of an act f; with $[f]_1$ (respectively, $[f]_2$) denoting the lower (respectively, upper) envelope.

Consider the preference relation $\succsim$ generated by the function

$$V(f) := \int_0^{\frac{1}{2}} \left(\frac{1}{2}[f]_1 + \frac{1}{2}[f]_2\right) d\mu + \int_{\frac{1}{2}}^1 \left(\frac{2}{3}[f]_1 + \frac{1}{3}[f]_2\right) d\mu\,. \tag{3}$$

Intuitively this is a "state-dependent" interval utility; however, for any ideal act f, since $[f]_1 = [f]_2$, V reduces to subjective expected utility with a linear Beroulli utility.

We show that $\succsim$ satisfies GP's Axioms A1–A6 which we list here for the convenience of the reader. To state them we employ the following notation: for any pair of acts f and g and any event $C \subset \Omega$, fCg denotes the act that agrees with f on C and with g on the complement of C. We also require the following definitions.

An event E is *ideal* if $[fEh \succsim gEh$ and $hEf \succsim hEg]$ implies $[fEh' \succsim gEh'$ and $h'Ef \succsim h'Eg]$ for all acts f, g, h, and h'. An event A is *null* if $fAh \sim gAh$ for all acts f, g and h. An event D is *diffuse* if $E \cap D \neq \emptyset \neq E \cap D^c$ for every non-null ideal event E. Let $\mathcal{E}$ (respectively, $\mathcal{N}$, $\mathcal{D}$) be the set of all ideal (respectively, null, diffuse) events. Let $\mathcal{F}^e$ denote the set of *ideal simple* acts.[1]

[1]A simple act is one that has a finite range; hence for any $f \in \mathcal{F}^e$ we have $f^{-1}(x) \in \mathcal{E}$ for all x.

As in GP, we say an event E is *left* (respectively, *right*) ideal if $fEh \succsim gEh$ implies $fEh' \succsim gEh'$ (respectively $hEf \succsim hEf$ implies $h'Ef \succsim h'Eg$). Let $\mathcal{E}^l$ and $\mathcal{E}^r$ be the collection of left and right ideal sets respectively. GP's Lemma B0 establishes $\mathcal{E} = \mathcal{E}^l \cap \mathcal{E}^r$.

In line with GP's use of notation, events E, E', E_i et cetera denote ideal events while events D, D', D_i denote diffuse events.

A1 $\succsim$ is complete and transitive.

A2 $f > g$ implies $f \succ g$.

A3 $yE \cap Dx \sim yE \cap D'x$ for all x, y, E, D and D'.

A4 If $y > x$ and $w > z$, then $yEx \succsim yE'x$ implies $wEz \succsim wE'z$

A5 If $f, g \in \mathcal{F}^e$ and $f \succ g$, then there is a partition $E_1, \cdots, E_n$ of Ω such that $\ell E_i f \succ mE_i g$ for all i.

A6 Let $g \succsim f_n \succsim h$ for all n. Then, (i) $f_n \in \mathcal{F}^e$ converges pointwise to f implies $g \succsim f \succsim h$. (ii) $f_n \in \mathcal{F}$ converges uniformly to f implies $g \succsim f \succsim h$.

To verify $\succsim$ satisfies the above six axioms, we utilize the fact that an event is deemed ideal by $\succsim$ if and only if it is measurable (that is, an element of $\mathcal{E}_\mu$).

Lemma 1. *For the relation $\succsim$ generated by V defined in* (3) *we have*

$$\mathcal{E} = \mathcal{E}_\mu .$$

Proof. See appendix.

Returning to the axioms we see each is verified as follows.

1. A1: Satisfied since $\succsim$ is generated by the real-valued function V defined in (3).

2. A2: Choose $f, g \in \mathcal{F}$ with $f > g$. If $f > g$ then $[f]_1(s) > [g]_1(s)$ and $[f]_2(s) > [g]_2(s)$ for all s. Applying Lemma 5:

$$V(f) - V(g) = \int_0^{0.5} \frac{1}{2}([f]_1 - [g]_1) + \frac{1}{2}([f]_2 - [g]_2) d\mu + \int_{0.5}^{1} \frac{2}{3}([f]_1 - [g]_1) + \frac{1}{3}([f]_2 - [g]_2) d\mu > 0 \,.$$

3. A3: Without loss of generality assume $x \leq y$, then $[yE \cap Dx]_1 = x$ and $[yE \cap Dx]_2 = (E : y, E^c : x)$ which does not depend on D, when D is an diffuse event, so A3 holds.

4. A4 and A5: Trivially satisfied since V is SEU for ideal acts.

5. A6 (ii): As in GP (2014), if f^n converges to f uniformly then $[f^n]_1$ (respectively, $[f^n]_2$) converges to $[f]_1$ (respectively, $[f]_2$) pointwise. Since f^n is (zeroth-order) dominated by the constant act yielding the maximal payoff m for certain, invoking A2 we have $|u \circ [m](s)| \geq |u \circ [f^n](s)|$ for all s. Therefore the integral of $V(f^n)$ converges to the integral of $V(f)$ by the Dominance Convergence Theorem. A6 (i) is true by a similar argument.

Since the preference relation $\succsim$ generated by (3) satisfies GP's axioms 1 – 6, it follows from GP's Theorem 1 that it should admit an EUU representation with prior μ.[2]

To demonstrate the preference relation $\succsim$ generated by (3) is in fact not EUU, first notice for the (ideal) event $E = [0, 1/2]$ in $\mathcal{E}_\mu$, since $\mu(E) = 1/2$ it follows that $mE\ell \sim \ell Em$. Now fix a diffuse event D and consider the pair of acts f and g in which $f(E \cap D) = \{m\}$ with $f(\omega) = \ell$ otherwise, and $g(E^c \cap D) = \{m\}$ with $g(\omega) = \ell$, otherwise. For GP's EUU maximizer we must have $f \sim g$ since

$$\begin{aligned} EUU(f) &= \mu(E)u(\ell, m) + (1 - \mu(E))\, u(\ell, \ell) \\ &= \frac{1}{2}u(\ell, m) + \frac{1}{2}u(\ell, \ell) \\ &= \mu(E)u(\ell, \ell) + (1 - \mu(E))\, u(\ell, m) = EUU(g) \,. \end{aligned}$$

[2] This follows since two linear representations of the same preference relation must be affine transformations of each other.

However, since $[f]_1 = [g]_1 = \ell$, $[f]_2 = mE\ell$, and $[g]_2 = \ell Em$, for the function V defined in (3), we have

$$V(f) = \ell + \frac{1}{4}(m - \ell) > \ell + \frac{1}{6}(m - \ell) = V(g),$$

that is, $f \succ g$. Thus, the preferences generated by V in (3) cannot be from the class of EUU maximizers.

3 Representation Theorem

We retain the first 5 axioms of GP and propose a modified A6(i) along with the original A6(ii) plus one extra new axiom A7. The former enables us to establish the set of ideal events is a σ-algebra while the latter ensures the constancy of conditional certainty equivalents of diffuse "bets" which rules out the (counter-)example from the previous section.

To see why we require a modification of GP's axiom A6(i), we point out in their proof of their Lemma B2 (which states the collection of ideal events is a σ-field), in the second paragraph on p28, they only establish

$$\begin{aligned} & [f \cup E_i h \succsim g \cup E_i h \text{ and } h \cup E_i f \succsim h \cup E_i g] \\ \implies & \left[f \cup E_i h' \succsim g \cup E_i h' \text{ and } h' \cup E_i f \succsim h' \cup E_i g\right] \text{ for all (ideal acts) } f, g, h, h' \in \mathcal{F}^e, \end{aligned}$$

and ***not*** for all (arbitrary acts) $f, g, h, h' \in \mathcal{F}$, as is required to establish an event is ideal. Now since their argument relies on their A6(i) which does not constrain non-ideal acts, without having first established the set of ideal events $\mathcal{E}$ is a σ-field, their earlier results (Lemmas A1 and A2 on p22) cannot establish the existence and uniqueness of the envelopes of acts which are needed to approximate non-ideal acts. Moreover, having failed to establish $\mathcal{E}$ is a $\sigma-$algebra, in turn means GP's Lemmas B4 and B5 on p28 cannot establish the existence and uniqueness of a countably additive probability measure μ on $\mathcal{E}$. We thus provide a stronger version of A6(i).

A6*

(i) Let $g \succsim fE^n f' \succsim h$ with $E_{n+1} \subset E_n$ for all n. Then $g \succsim f{\cap}E^n f' \succsim h$.

(ii) Let $g \succsim f_n \succsim h$ for all n. Then $f_n \in \mathcal{F}$ converges uniformly to f implies $g \succsim f \succsim h$.

A7 $xDy \succsim z$ implies $(xDy)Ef \succsim zEf$ for all x, y, z and f.

Theorem 1. *The relation $\succsim$ satisfies A1 – A5, A6*, and A7 if and only if $\succsim$ admits an EUU representation.*

Proof. **Outline of sufficiency Proof:**
Following the outline of GP's proof, we note that by restricting attention to ideal acts, Axioms A1 – A5 plus our axiom A6* yield a standard expected utility representation with a countably additive probability measure μ and a continuous Bernoulli utility $v\colon X \to \mathbb{R}$.

We turn now to general acts.

(i) The first step is to prove that $\mathcal{E}$ (the set of ideal events) is a σ-algebra. It uses our revised continuity axiom A6* and Facts 1 and 2 and other parts of Lemma B2 from GP's Appendix B. Notice that the new continuity axiom is more than a technical tweak in this set-up; it ensures that $\mathcal{E}$ is not only an algebra but also a $\sigma-$algebra, as this guarantees the existence of the associated envelopes, and establishing the interval utility is well-defined also relies on the envelopes being well-defined.

(ii) The second step is to prove the existence and uniqueness (up to a measure zero set) of the envelope $[f]$. This part follows the analysis and results of GP's Appendix A. We highlight that this step needs to use the result from step (i).

(iii) The third step is to prove that an EUU functional constructed using the prior μ from the SEU representation of the restriction of the preferences to ideal acts and an interval utility defined by setting $u(x\,,y) := v(z)$, where for any $x \leqslant y$, z is chosen such that $yDz \sim z$ for some diffuse D, represents $\succsim$. Axioms A2 and A6* together imply that $z \in [x\,,y]$ and A3 means it does not matter which diffuse event D is used, thereby ensuring this interval utility well-defined. It uses Lemma B3-B8, a modified version of Lemma B9, and Lemma B10.

As we noted in the introduction above, the expected uncertainty utility of an act f is intuitively the subjective expected utility of its envelope $[f]$: $EUU(f) = SEU([f])$. That is, Savage's P1 to P5 defined on the induced preferences over envelopes must be necessary, and our axioms must be sufficient to imply that the induced preferences over envelopes satisfy P1 to P5. Intuitively A1 implies P1; the definition of ideal events implies P2; A7 implies P3; A7 and A4 together imply P4; A2 implies P5.

The detailed sufficiency proof is presented here. The first few parts of GP's Lemma B2's proof demonstrate that $\mathcal{E}$ is an algebra. The remainder of step 1 is to prove that, with our revised continuity axiom, $\mathcal{E}$ is a $\sigma-$algebra but to do so we first require the following.

Lemma 2. *A null event $\hat{E}$ is ideal.*

Proof. A null event $\hat{E}$ is left ideal by definition. Then we will show a null event $\hat{E}$ is right ideal. Let $h\hat{E}f \succsim h\hat{E}g$. By definition of null event: $h\hat{E}f \sim h'\hat{E}f$ and $h\hat{E}g \sim h'\hat{E}g$. By transitivity, $h'\hat{E}f \succsim h'\hat{E}g$, which finishes the proof. □

Lemma 3. *If $\{E_n\}$ is a sequence of null events, $\cap E_n$ is null.*

Proof. Assume *per contra*, $\cap E_n$ is non-null. There are f, g, h such that $f \cap E_n h \succ g \cap E_n h$, that is, $(f \cap E_n h)E_m h \succ (g \cap E_n h)E_m h$, which contradicts to the fact that E_m is null and so $\cap E_n$ is null. □

Lemma 4. *$\mathcal{E}$ is a $\sigma-$algebra.*

Proof. As $\mathcal{E}$ is an algebra already, we need only show the countable union of ideal events is ideal. We proceed by establishing the countable intersection of ideal events is ideal. Let $E^n \in \mathcal{E}$ and $E^{n+1} \subset E^n$. We first show $\cap E^n \in \mathcal{E}^l$. Assume *per contra*, there are f, g, h, h' such that $f \cap E^n h \succsim g \cap E^n h$ and $g \cap E^n h' \succ f \cap E^n h'$. By $g \cap E^n h' \succ f \cap E^n h'$, there is N such that for all $n > N$, $gE^n h' \succ fE^n h'$ and so $gE^n h \succ fE^n h$ and by Axiom 6(i), $g \cap E^n h \succsim f \cap E^n h$.

We have $f \cap E^n h \sim g \cap E^n h$ and $g \cap E^n h' \succ f \cap E^n h'$. Since $f \cap E^n h = (f \cap E^n h)E^n h \sim g \cap E^n h = (g \cap E^n h)E^n h$, then $(f \cap E^n h)E^n h' \sim (g \cap E^n h)E^n h'$ and so $(f \cap E^n h) \cap E^n h' \sim (g \cap E^n h) \cap E^n h'$ by Axiom 6(i), that is, $f \cap E^n h' \sim g \cap E^n h'$, which gives us a contradiction and $\cap E^n \in \mathcal{E}^l$.

We next show $\cap E^n \in \mathcal{E}^r$, that is, $h \cap E^n f \succsim h \cap E^n g$ implies $h' \cap E^n f \succsim h' \cap E^n g$ for all f, g, h, h'. It is enough to show that $(\cap E^n)^c$ is left ideal. This part uses Theorem 1 of Gorman (1968). To use Gorman's theorem, first note that $\succsim$ is a weak order so Gorman's assumption (0) is satisfied. Then, let $A = \cap E_n$ and $B = \Omega$, and note that both event A and B are left ideal, so the required assumption (i) of Gorman's theorem is satisfied.[3] Then, there will be three cases to be discussed: case 1, $B - A$ is a null set; case 2, $B - A$ is essential but not strictly essential; case 3, $B - A$ is strictly essential. We use Gorman's definition for event essentiality.

Case 1: if $B - A$ is a null set, then $B - A = (\cap E^n)^c$ must be ideal by Lemma 2.

Case 2: if $C = B - A$ is essential but not strictly essential, there exists h such that $fCh \sim f'Ch$ for all f, f'. That is, $h \cap E_n f \sim h \cap E_n f'$ for all f, f', which implies $h' E_n f \sim h' E_n f'$ for all f, f' with $h'(s) = h(s)$ if $s \in \cap E_n$, so E_n is not strictly essential for each E_n.

Each event E_n is ideal, so E_n is either strictly essential or null. It follows that all E_n must be null. Then $\cap E^n$ is also null by Lemma 3. Hence $\cap E^n$ is ideal by Lemma 2, and $(\cap E^n)^c$ ideal by GP's Fact 1(ii).

Case 3: $B - A$ is strictly essential. In this case, all required assumptions of Gorman's theorem is satisfied and $B - A = (\cap E^n)^c$ must be left ideal.

The event $(\cap E^n)^c$ being left ideal implies that $\cap E^n$ is right ideal. Then $\cap E^n$ is both right, left ideal, and therefore ideal.

Finally, suppose $E_n \in \mathcal{E}$ for each n, then $E_n^c \in \mathcal{E}$ because $\mathcal{E}$ is an algebra. Then we have $\cap E_n^c \in \mathcal{E}$. By De Morgan's Law on countable union, $\cap E_n^c = (\cup E_n)^c \in \mathcal{E}$. Since $\mathcal{E}$ is an algebra, $\cup E_n \in \mathcal{E}$, which finishes the proof. $\square$

Having established $\mathcal{E}$ a $\sigma-$algebra, we can now apply GP's Lemmas A1, A2, and 1.

For the third step, GP's Lemma B3 ensures the existence of an SEU that represents the restriction of the preference relation to ideal acts. GP's Lemma B4 ensures that the probability measure μ of SEU characterized in B3 is also a prior. GP's Lemma B5 ensures that the utility function of SEU over ideal acts must be increasing and continuous. GP's Lemma B6 ensures that the set of all diffuse acts generated by the preference

[3] Gorman (1968) uses the name "separable event" for left ideal event in the sense of GP.

is same as the set of all diffuse acts generated by μ, and the certainty equivalent of xDy is unique. B6 ensures the interval utility is well-defined and GP's Lemma B8 ensures the monotonicity and continuity of the interval utility u. It is now enough to modify slightly GP's Lemma B9:

Lemma B9* The function V defined by (2) represents the restriction of $\succsim$ to $\mathcal{F}_0$.

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be the range of f, let $A_i = f^{-1}(x_i)$, and let $\{E_J^*(f)\}$ be an ideal split of $\{A_i\}$. Lemma A2 implies that $\{E_J^*(f)\}$ exists and is unique up to measure zero. Let $N^+(f) = \{J \mid \mu(E_J^*(f)) > 0 \text{ and } |J| > 1\}$ and $H_n = \{f \in \mathcal{F}_0 \mid n = |N^+(f)|\}$. The proof is by induction on H_n. Note that for $f \in H_0$, $V(f) = \int_X v(z)\mu(f^{-1}(z))dz = v(x)$ for x such that $\mu(\{x = f\}) = 1$. Hence, by Lemma B3(ii), V represents the restriction of φ to H_0. Suppose V represents the restriction of φ to H_n and choose $f \in H_{n+1}$. Define h_f as follows: if $f \in H_n$, then $h_f = f$; otherwise, choose $E_J^*(f)$ such that $|J| > 1$ and $\mu(E_J^*(f)) > 0$, choose $D \in \mathcal{D}$, and define f^* as follows:

$$f^*(\omega) = \begin{cases} f(\omega) & \text{if } \omega \notin E_J^*(f), \\ \max\{f(\omega), E_J^*(f)\} & \text{if } \omega \in D \cap E_J^*(f), \\ \min\{f(\omega), E_J^*(f)\} & \text{if } \omega \in D^c \cap E_J^*(f). \end{cases}$$

By Lemma B7 and Axiom A3, $f^* \sim f$. Next, choose z such that $u(x,y) = v(z)$, where recall z is the certainty equivalent of xDy, for some diffuse D. Let $h_f(\omega) = f^*(\omega)$ for all $\omega \notin E_J^*(f)$ and $h_f(\omega) = z$ for all $\omega \in E_J^*(f)$. Axiom A7 ensures the constancy of the conditional certainty equivalents of diffuse acts: $h_f \sim f^* \sim f$. Notice that $h_f \in H_n$ and, by construction, $V(h_f) = V(f^*)$. By Lemma A1, $[f^*] = [f]$ and, therefore, $V(f^*) = V(f)$. Thus, $V(f) = V(h_f)$ for some $h_f \in H_n$ such that $h_f \sim f$. Then, the induction hypothesis implies that V represents φ on H_{n+1}. □

As was the case in GP, the extension to all acts can be obtained using Axiom 6*(ii) and follows familiar arguments in GP's Lemma B10.

Necessity Proof: Fix V an EUU functional. GP prove that the relation $\succsim$ generated by V satisfies A1-A5 and A6(ii). The rest of the proof will establish this relation also satisfies A6*(i) and A7. When $n \to +\infty$, $E^n \to \cap E^n$. Without loss of generality assume $f \precsim f'$ when restricted to the event $E^n \setminus \cap E^n$. Because $\ell \leq f(\omega) \leq m$, we have $V(f \cap E^n f') - V(fE^n f') \leq \int_{E^n \setminus \cap E^n} \big(u(m,m) - u(\ell,\ell)\big) d\mu = \big(u(m,m) - u(\ell,\ell)\big)\mu(E^n \setminus \cap E^n)$. Since μ is a probability measure on a $\sigma-$ algebra, $E^n \to \cap E^n$ implies $\mu(E^n) \to \mu(\cap E^n)$ and $\mu(E^n \setminus \cap E^n) \to 0$. Therefore $V(f \cap E^n f') - V(fE^n f') \to 0$ as $n \to +\infty$.

For A7, suppose x, y, c are constant acts and$x_D y \sim c$. We want to show that $(x_D y)Ef \sim cEf$ for all E, f. When $x = y$ the axiom trivially holds. Without loss of generality, suppose $x < y$, then the interval utility $u(x,y) = u(c,c)$. Then $\int_E u(x,y) d\mu = \int_E u(c,c) d\mu$, $\int_E u(x,y) d\mu + \int_{E^c} f d\mu = \int_E u(c,c) d\mu + \int_{E^c} f d\mu$, and $V((x_D y)Ef) = V(cEf)$.

□

Appendix

Proof of Lemma 1

$\mathcal{E}_\mu \subseteq \mathcal{E}$.

For any $E \in \mathcal{E}_\mu$ it is immediate from the representation in (3) that it satisfies the definition of an ideal event.

$\mathcal{E} \subseteq \mathcal{E}_\mu$.

It suffices to show that any event not in $\mathcal{E}_\mu$ will generate a violation of P2 for some acts. Fix $A \notin \mathcal{E}_\mu$ and let $E^{\{1\}}$ (respectively, $E^{\{2\}}$) denote the largest measurable subset of (respectively, the complement of) A.[4] Following GP, we consider the same events B_{11}, B_{12},B_{21} and B_{22}, all of which do not contain any element in $\mathcal{E}_\mu$ and for which $A = E^{\{1\}} \cup B_{11} \cup B_{12}$ and $A^c = E^{\{2\}} \cup B_{21} \cup B_{22}$. Let $E^{\{1,2\}} = B_{11} \cup B_{12} \cup B_{21} \cup B_{22} \in \mathcal{E}_\mu$.

For $x_1 < x_2 < x_3 \in X$ with $x_3 - x_2 > x_2 - x_1$, let $g = x_1$ and let

$$f = \begin{cases} x_3, & \text{if } x \in B_{11}, \\ x_2, & \text{if } x \in B_{12}, \\ x_1, & \text{otherwise.} \end{cases} \qquad h = \begin{cases} x_3, & \text{if } x \in B_{11}, \\ x_1, & \text{if } x \in B_{12}, \\ x_1, & \text{otherwise.} \end{cases} \qquad h' = \begin{cases} x_2, & \text{if } x \in B_{11}, \\ x_2, & \text{if } x \in B_{12}, \\ x_1, & \text{otherwise.} \end{cases}$$

So we have

$$[f_A h]_1 = x_1 \text{ and } [f_A h]_2 = x_3 E^{\{1,2\}} x_1$$

[4] Since μ is countably additive such events exist and are unique up to zero measure sets.

$$[f_A h']_1 = x_2 E^{\{1,2\}} x_1 \text{ and } [f_A h']_2 = x_3 E^{\{1,2\}} x_1$$

$$[g_A h]_1 = x_1 \text{ and } [g_A h]_2 = x_3 E^{\{1,2\}} x_1$$

$$[g_A h']_1 = x_1 \text{ and } [g_A h']_2 = x_2 E^{\{1,2\}} x_1$$

By the representation, we have $f_A h \sim g_A h$ but $f_A h' \succ g_A h'$, and so A is not a left ideal and A is not an ideal event. □

We use the following lemma in the sufficiency proof.

Lemma 5. *Suppose that $f(\omega) > 0$ for all ω and $\mu(\Omega) > 0$, then $\int_\Omega f d\mu > 0$*

Proof. Define $\Omega_k = \Omega \cap \{f > \frac{1}{k}\}$. It follows that $\Omega = \cup \Omega_k$. Assume for each k, Ω_k is measure zero; then, Ω is measure zero, which contradicts to the fact that μ is a probability measure with $\mu(\Omega) = 1$. Therefore $\mu(\Omega_k) > 0$ for at least one k. So $\int_\Omega f \geq \int_{\Omega_k} f \geq \frac{1}{k}\mu(\Omega_k) > 0$. □

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